Signals & Systems

Index

Basics of Signals & Systems

Properties of Signals

A signal can be classified as periodic or aperiodic; discrete or continuous time; discrete of continuous-valued; or as a power or energy signal. The following defines each of these terms. In addition, the signal-to-noise ratio of a signal corrupted by noise is defined.

Periodic / Aperiodic:

A periodic signal repeats itself at regular intervals. In general, any signal *x*(*t*) for which

x(t) = x(t+T)

for all *t* is said to be *periodic*.

The fundamental period of the signal is the minimum positive, non-zero value of *T* for which above equation is satisfied. If a signal is not periodic, then it is *aperiodic*.

Symmetric / Asymmetric:

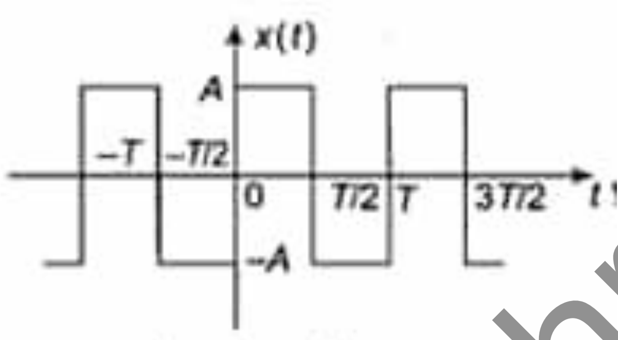
There are two types of signal symmetry: odd and even. A signal x(t) has odd symmetry if and only if x(-t) = -x(t) for all t. It has even symmetry if and only if x(-t) = x(t).

Continuous and Discrete Signals and Systems

A continuous signal is a mathematical function of an independent variable, which represents a set of real numbers. It is required that signals are uniquely defined in except for a finite number of points.

- A continuous time signal is one which is defined for all values of time. A continuous time signal does not need to be continuous (in the mathematical sense) at all points in time. A continuous-time signal contains values for all real numbers along the X-axis. It is denoted by x(t).
- Basically, the Signals are detectable quantities which are used to convey some information about time-varying physical phenomena. some examples of signals are human speech, temperature, pressure, and stock prices.
- Electrical signals, normally expressed in the form of voltage or current waveforms, they are some of the easiest signals to generate and process.

Example: A rectangular wave is discontinuous at several points but it is continuous time signal.



A rectangular wave

Discrete / Continuous-Time Signals:

A continuous time signal is defined for all values of *t*. A discrete time signal is only defined for discrete values of $t = ..., t_{-1}, t_0, t_1, ..., t_n, t_{n+1}, t_{n+2}, ...$ It is uncommon for the spacing between t_n and t_{n+1} to change with *n*. The spacing is most often some constant value referred to as the sampling rate,

$T_s = t_{n+1} - t_n$.

It is convenient to express discrete time signals as $x(nT_s)=x[n]$.

That is, if $x(t)$ is a continuous-time signal, then $x[n]$ can be considered as the n^{th} sample of $x(t)$.

Sampling of a continuous-time signal $x(t)$ to yield the discrete-time signal $x[n]$ is an important step in the process of digitizing a signal.

Energy and Power Signal:

When the strength of a signal is measured, it is usually the signal power or signal energy that is of interest.

The signal power of *x*(*t*) is defined as

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

and the signal energy as

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- A signal for which P_x is finite and non-zero is known as a *power signal*.
- A signal for which E_x is finite and non-zero is known as an *energy signal*.
- P_x is also known as the *mean-square* value of the signal.
- Signal power is often expressed in the units of decibels (dB).
- The decibel is defined as

$$P_{xdB} = 10\log\left(\frac{P_x}{P_0}\right)$$

- where P_0 is a reference power level, usually equal to one squared SI unit of the signal.
- For example if the signal is a voltage then the P_0 is equal to one square Volt.
- A Signal can be Energy Signal or a Power Signal but it can not be both. Also a signal can be neither a Energy nor a Power Signal.
- As an example, the sinusoidal test signal of amplitude *A*,

x(t)=Asin(ωt)

has energy E_x that tends to infinity and power ,

$$P_x = \frac{1}{2}A^2$$

or in decibels (dB): 20log(A)-3

The signal is thus a power signal.

Signal to Noise Ratio:

Any measurement of a signal necessarily contains some random noise in addition to the signal. In the case of additive noise, the measurement is

x(t)=s(t)+n(t)

where $s(t)$ is the signal component and $n(t)$ is the noise component.

The signal to noise ratio is defined as

$$SNR_x = \frac{P_s}{P_n}$$

or in decibels,

$$SNR_x = 10\log\left(\frac{P_s}{P_n}\right)$$

The signal to noise ratio is an indication of how much noise is contained in a measurement.

Standard Continuous Time Signals

- **Impulse Signal**

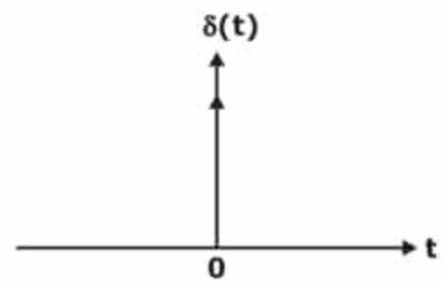
$$\delta(t) = \begin{cases} \infty ; t = 0 \\ 0 ; t \neq 0 \end{cases}$$

where ∞ is the height of impulse signal having unit area.

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = A$$

When A = 1 (unit impulse Area)



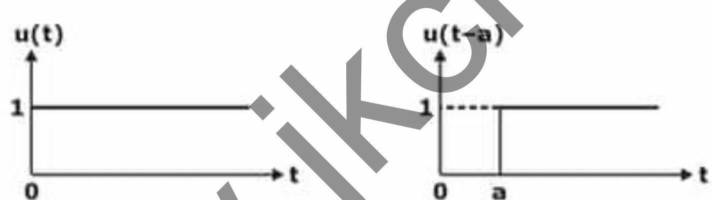
Unit impulse function

- **Step Signal**

$$x(t) = \begin{cases} A\,;\, t \geq 0 \\ 0\,;\, t = 0 \end{cases}$$

Unit Step Signal if A =1,

$$x(t) = u(t) = \begin{cases} 1\,;\, t > 0 \\ 0\,;\, t < 0 \end{cases}$$



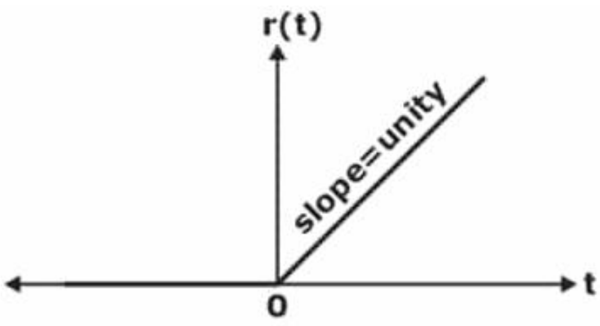
(a) unit step function (b) Shifted Unit Step Function

- **Ramp Signal**

$$x(t) = \begin{cases} At\,;\, t \geq 0 \\ 0\,;\, t = 0 \end{cases}$$

Unit Ramp Signal (A=1)

$$x(t) = r(t) = \begin{cases} t\,;\, t \geq 0 \\ 0\,;\, t < 0 \end{cases}$$



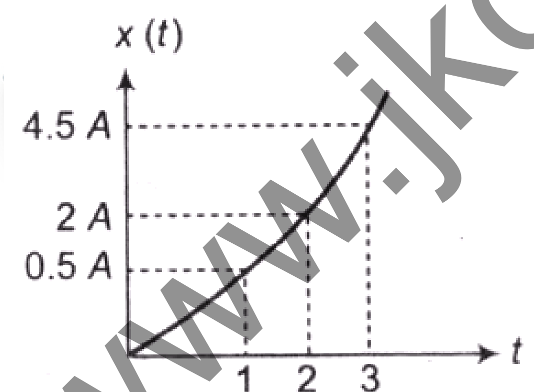
Unit Ramp Function

- **Parabolic Signal**

$$x(t) = \begin{cases} \dfrac{At^2}{2}\ ; t \geq 0 \\ 0\ ; t < 0 \end{cases}$$

Unit Parabolic Signal when A = 1,

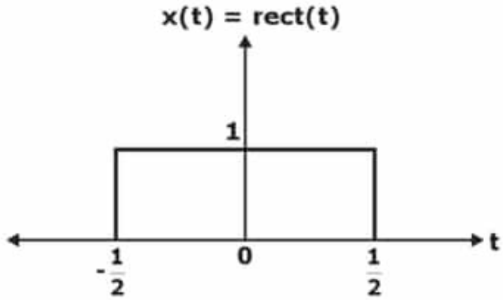
$$x(t) = \begin{cases} \dfrac{t^2}{2}\ ; t \geq 0 \\ 0\ ; t < 0 \end{cases}$$



Parabolic function

- **Unit Pulse Signal**

$$\begin{aligned} x(t) &= \pi(t) \\ &= u(t + 1/2) - u(t - 1/2) \end{aligned}$$



Unit Rectangular Pulse Function

Sinusoidal Signal

- **Co-sinusoidal Signal:**

$x.(t) = A\cos(\omega_0 t + \phi)$

Where, ω_0 is the angular frequency in rad/sec

f_0 = frequency in cycle/sec or Hz

T = time period in second

When

$\phi = 0,\ x(t) = A\cos\omega_0 t$

When ϕ = positive,

$x(t) = A\cos(\omega_0 t + \phi)$

When ϕ = negative,

$x(t) = A\cos(\omega_0 t - \phi)$

Sinusoidal Signal:

$x(t) = A\sin(\omega_0 t + \phi)$

Where,

$\omega_0 = 2\pi t_0 = \frac{2\pi}{T} =$

Angular frequency in red/sec

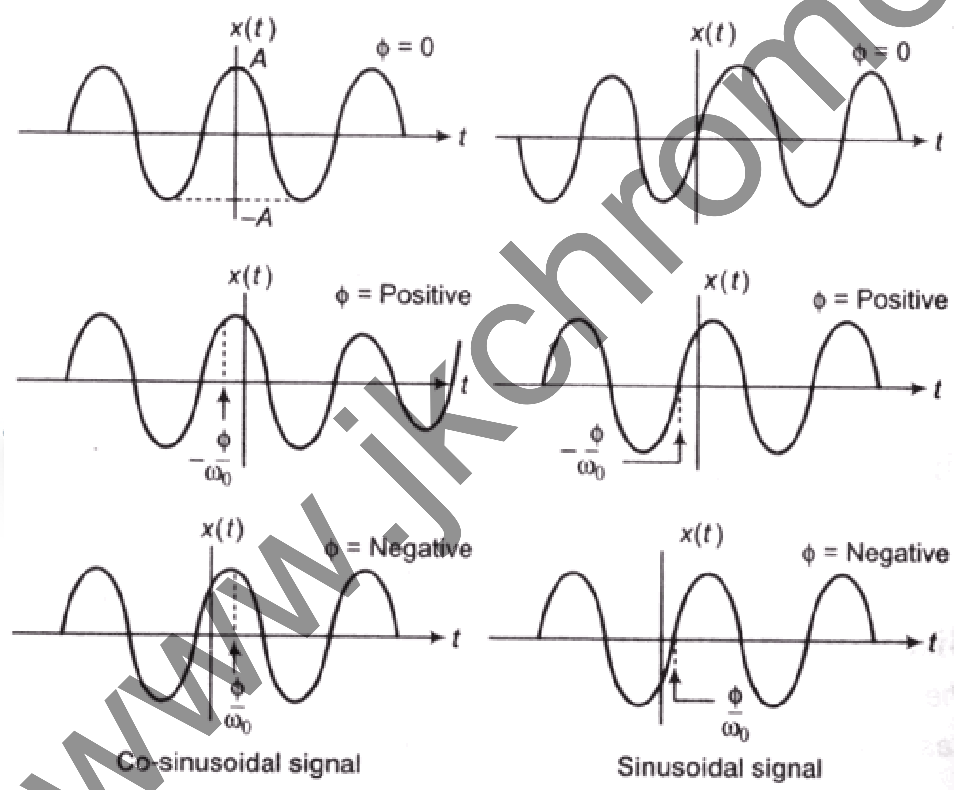
f_0 = frequency in cycle/sec or Hz

T = time period in second

When $\phi = 0, x(t) = A \sin(\omega_0 t)$

When ϕ = positive, $x(t) = A \sin(\omega_0 t + \phi)$

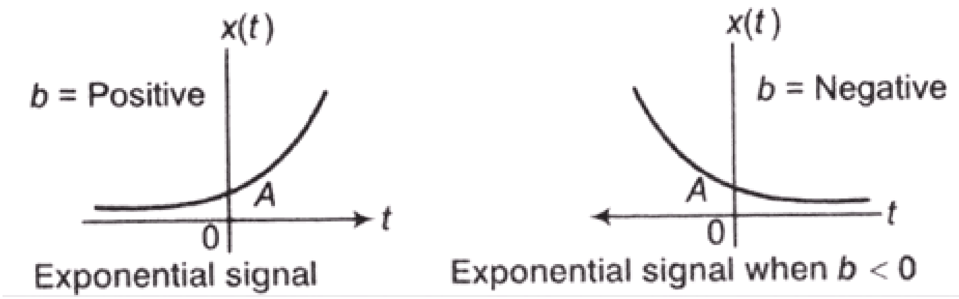
When ϕ = negative, $x(t) = A \sin(\omega_0 t - \phi)$



Exponential Signal:

- **Real Exponential Signal**

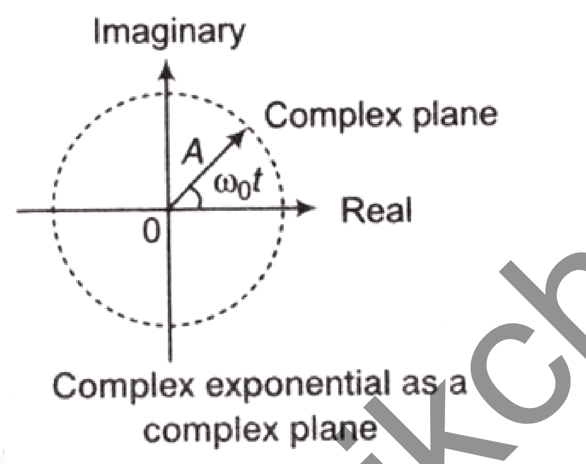
$x(t) = A\,e^{bt}$; where, A and b are real.



- **Complex Exponential signal**

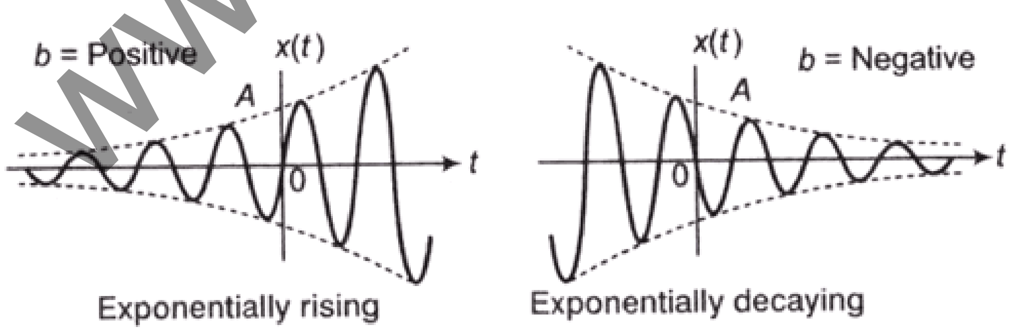
$$x(t) = A\, e^{j\omega_0 t}$$

The complex exponential signal can be represented in a complex plane by a rotating vector, which rotates with a constant angular velocity of ω_0 red/sec.



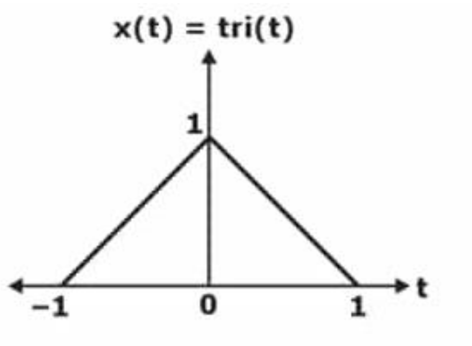
- **Exponentially Rising/Decaying Sinusoidal Signal**

$$x(t) = A\, e^{bt} \sin \omega_0 t$$



- **Triangular Pulse Signal**

$$x(t) = \triangle a(t) = \begin{cases} 1 - \dfrac{|t|}{a}; |t| \leq a \\ 0 ; |t| > a \end{cases}$$



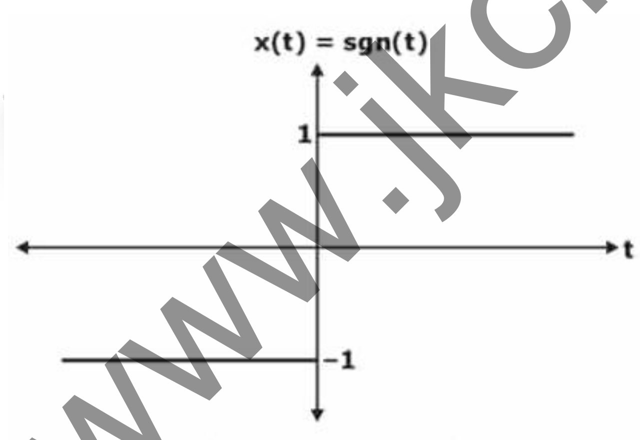
Unit Triangular Function

- **Signum Signal**

$$x(t) = Sgn(t) = \begin{cases} 1 \ ; \ t > 0 \\ -1 ; \ t < 0 \end{cases}$$

$$Sgn(t) = 2u(t) - 1$$

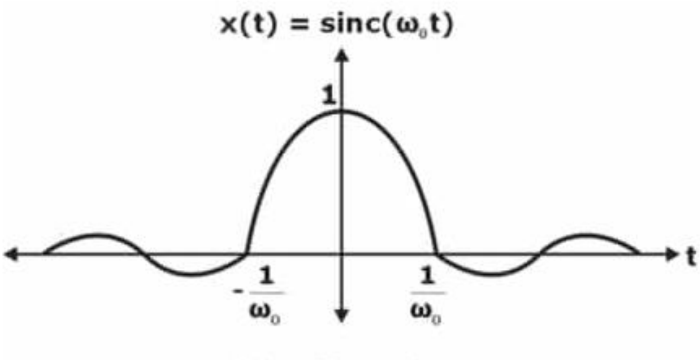
$$Sgn(t) = u(t) - u(-t)$$



Unit Signum Function

- **SinC Signal**

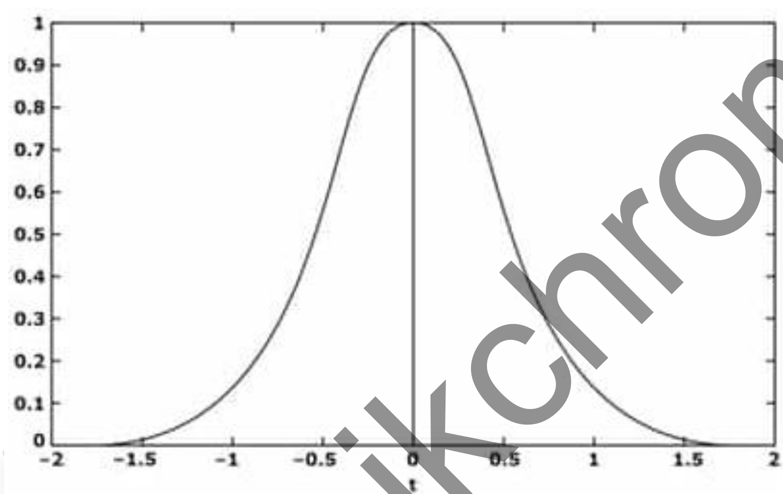
$$x(t) = \sin C(t) = \frac{\sin t}{t}; \infty < t < \infty$$



Sinc Function

- **Gaussian Signal**

$$x(t) = g_a(t) = e^{-a^2t^2}; \infty < t < \infty$$



Gaussian function

Important points:

- The sinusoidal and complex exponential signals are always periodic.
- The sum of two periodic signals is also periodic if the ratio of their fundamental periods is a rational number.
- Ideally, an impulse signal is a signal with infinite magnitude and zero duration.
- Practically, an impulse signal is a signal with large magnitude and short duration.

Classification of Continuous Time Signal: The continuous time signal can be classified as

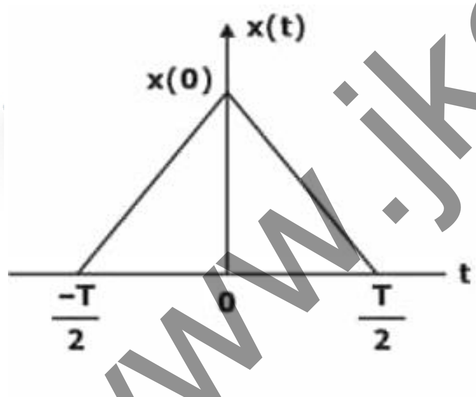
1. **Deterministic and Non-deterministic Signals:**
   - The signal that can be completely specified by a mathematical equation is called a deterministic signal. The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.
   - The signal whose characteristics are random in nature is called a non-deterministic signal. The noise signal from various sources like electronic amplifiers, oscillator etc., are examples of non-deterministic signals.
   - Periodic and Non-periodic Signals
   - A periodic signal will have a definite pattern that repeats again and again over a certain period of time.

$x(t+T) = x(t)$

2. **Symmetric (even) and Anti-symmetric (odd) Signals**

When a signal exhibits symmetry with respect to t = 0, then it is called an **even signal.**

$x(-t) = x(t)$



Even Signal

When a signal exhibits anti-symmetry with respect to t = 0, then it is called an **odd signal.**

$x(-t) = -x(t)$

Let

$X(t) = X_e(t) + X_0(t)$

Where,

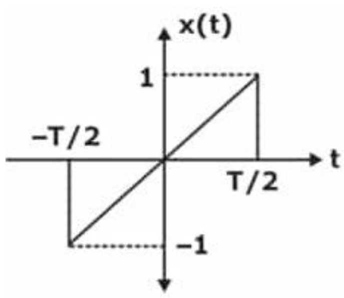
$X_e(t) =$

even part of

$X_0(t) =$

odd part of $X(t)$

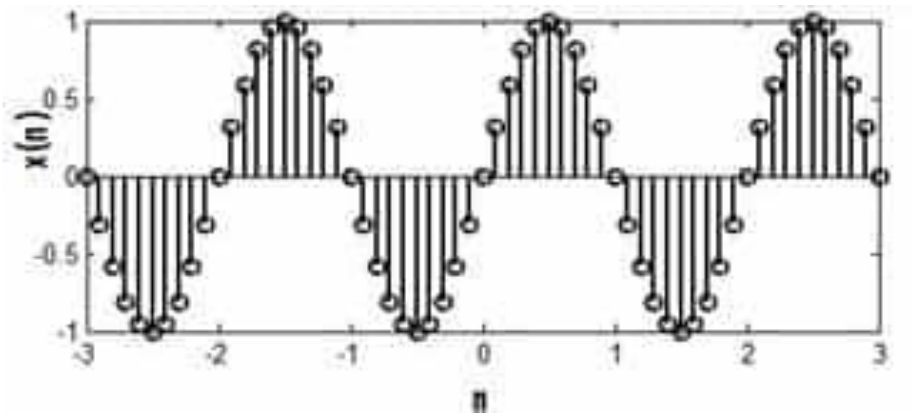
$$X_e(t) = \frac{1}{2}\left[X(t) + X(-t) \right]$$
$$X_0(t) = \frac{1}{2}\left[X(t) - X(-t) \right]$$



Odd signal

Discrete-Time Signals

The discrete signal is a function of a discrete independent variable. In a discrete time signal, the value of discrete time signal and the independent variable time are discrete. The digital signal is same as discrete signal except that the magnitude of the signal is quantized. Basically, discrete time signals can be obtained by sampling a continuous-time signal. It is denoted as x(n).

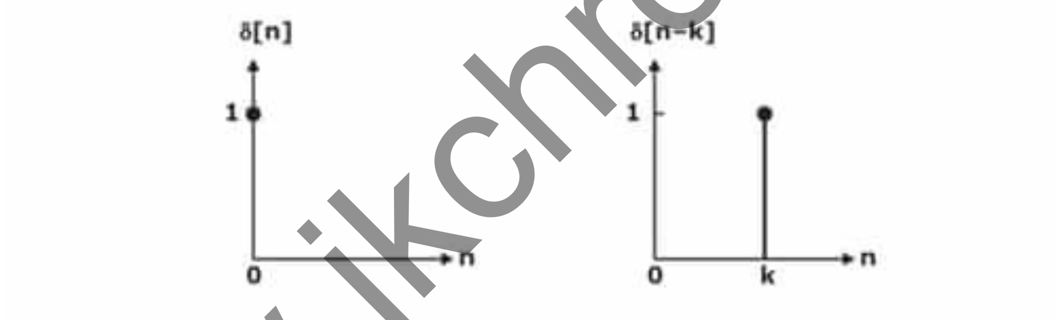


Standard Discrete Time Signals

- **Digital Impulse Signal or Unit Sample Sequence**

Impulse signal,

$$\delta(n)=\begin{cases}1; & n=0\\ 0; & n\neq 0\end{cases}$$



(a) DT Unit Impulse Function (b) DT Shifted Unit Impulse Function

- **Unit Step Signal**

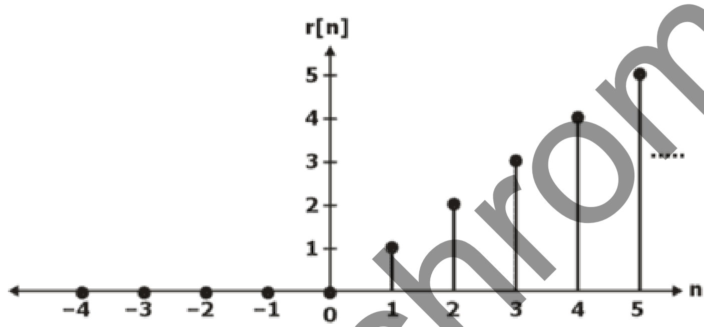
$$u(n)=\begin{cases}1; & n\geq 0\\ 0; & n<0\end{cases}$$

(a) DT Unit Impulse Function,(b) Shifted DT Unit Impulse Function

- **Ramp Signal**

Ramp signal,

$$u_r(n) = \begin{cases} n; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

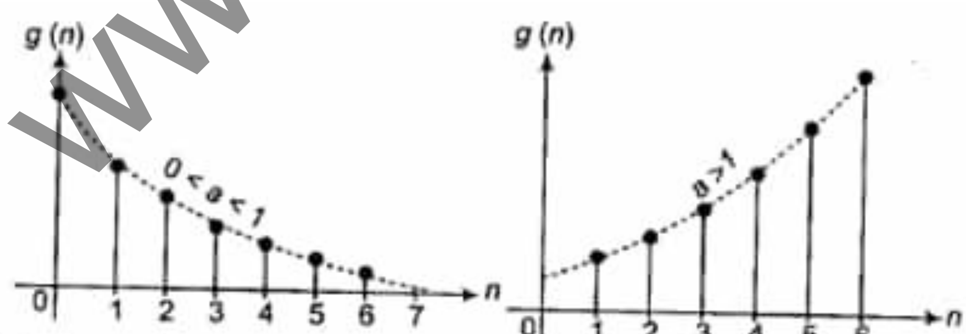


DT unit Ramp Function

- **Exponential Signal**

$$g(n) = \begin{cases} a^n; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

Exponential Signal,

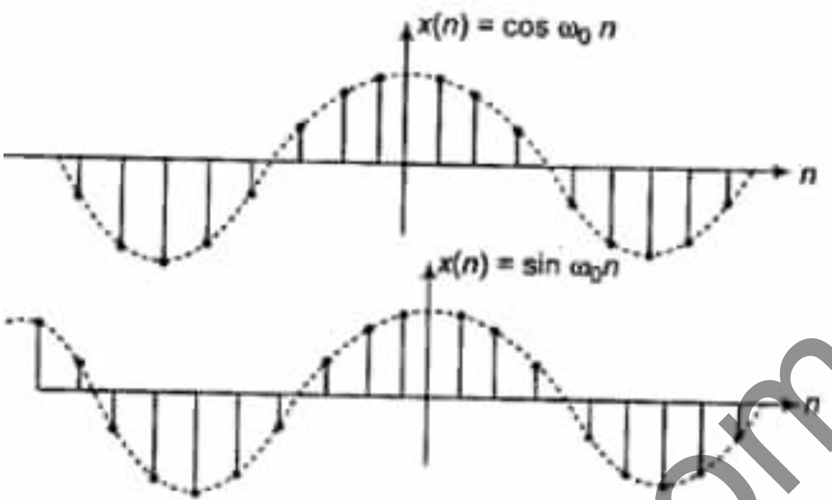


(a) Decreasing exponential signal (b) Increasing exponential signal

- **Discrete Time Sinusoidal Signal**

$x[n] = A\cos(\omega_0 n + \theta)$; For n in the range $-\infty < n < \infty$

$x[n] = A\sin(\omega_0 n + \theta)$; For n in the range $-\infty < n < \infty$

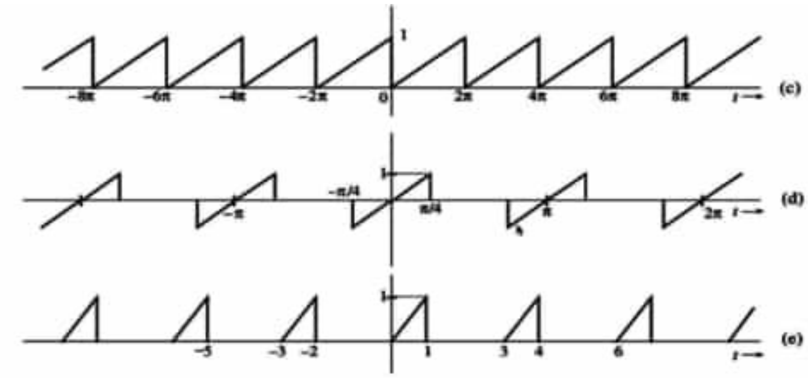


- A discrete-time sinusoid is periodic only if its frequency is a rational number.
- Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.

Operations in Continuous Time Signals:

Periodic & Non-Periodic Signals:

- A signal is a periodic signal if it completes a pattern within a measurable time frame, called a period and repeats that pattern over identical subsequent periods.
- The **period** is the smallest value of *T* satisfying *g(t + T) = g(t)* for all *t*. The period is defined so because if *g(t + T) = g(t)* for all *t*, it can be verified that *g(t + T') = g(t)* for all *t* where *T' = 2T, 3T, 4T, ...* In essence, it's the smallest amount of time it takes for the function to repeat itself. If the period of a function is finite, the function is called "periodic".
- Functions that never repeat themselves have an infinite period, and are known as "aperiodic functions".

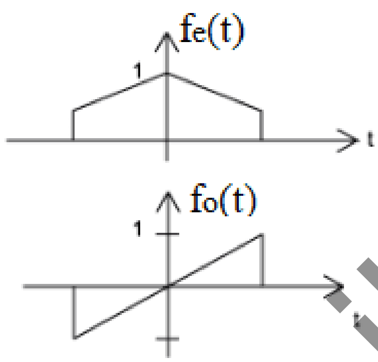


Even & Odd Signals:

A function even function if it is symmetric about the y-axis. While, A signal is odd if it is inversely symmetrical about the *y*-axis.

Even Signal, f(x) = f(-x)

Odd Signal, f(x) = - f(-x)



Note: Some functions are neither even nor odd. These functions can be written as a sum of even and odd functions. A function *f(x)* can be expressed in terms of sum of an odd function and an even function.

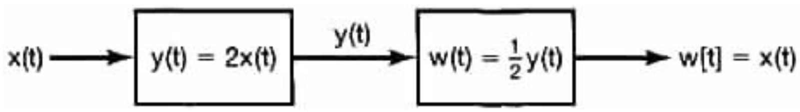
$$f(x) = 1/2\{f_e(x) + f_o(-x)\} + 1/2\{f_e(x) - f_o(-x)\}$$

Invertibility and Inverse Systems:

A system is invertible if distinct inputs results distinct outputs. As shown in the figure for the continuous-time case, if a system is invertible, then an inverse system exists that, when cascaded with the original system, results an output w(t) equal to the input x(t) to the first system.

An example of an invertible continuous-time system is **y(t) = 2x(t)**,

for which the inverse system is **w(t) = 1/2 y(t)**



Causal System:

A system is causal if the output depends only on the input at the present time and in the past. Such systems are often referred as non anticipative, as the system output does not anticipate future values of the input. Similarly, if two inputs to a causal system are identical up to some point in time t_o or n_o the corresponding outputs must also be equal up to this same time.

$y_1(t) = 2x(t) + x(t-1) + [x(t)]^2 \Rightarrow$ **Causal Signal**

$y_1(t) = 2x(t) + x(t-1) + [x(t+2)] \Rightarrow$ **Non-Causal Signal**

Homogeneity (Scaling):

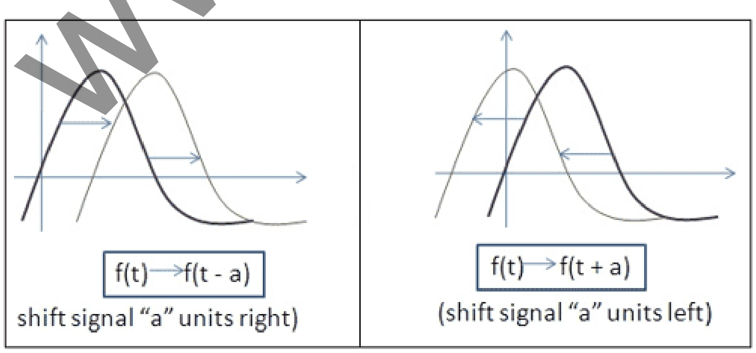
A system is said to be homogeneous if, for any input signal X(t), i.e. When the input signal is scaled, the output signal is scaled by the same factor.

$$X(t) \rightarrow S \rightarrow Y(s) \Rightarrow \alpha X(t) \rightarrow S \rightarrow \alpha Y(s)$$

Time-Shifting / Time Reversal / Time Scaling:

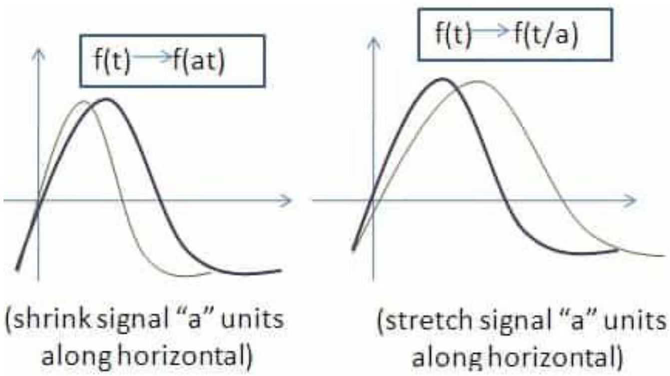
Time-Shifting

Time Shifting can be understood as shifting the signal in time. When a constant is added to the time, we obtain the advanced signal, & when we decrease the time, we get the delayed signal.



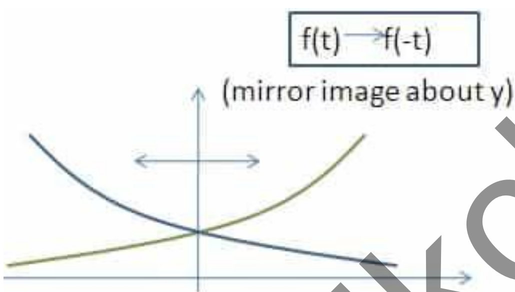
Time Scaling:

Due to the scaling in time the output Signal may shrink or stretch it depends on the numerical value of scaling factor.



Time Inversion:

Time Inversion referred as flipping the signal about the y-axis.



L.T.I. Systems

Linear Time-Invariant System:

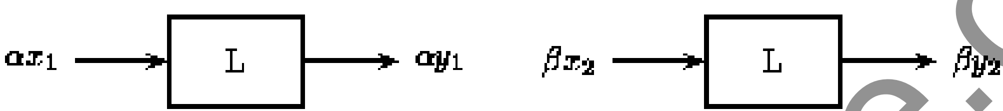
Linear time-invariant systems (LTI systems) are a class of systems used in signals and systems that are both linear and time-invariant. Linear systems are systems whose outputs for a linear combination of inputs are the same as a linear combination of individual responses to those inputs. Time-invariant systems are systems where the output does not depend on *when* input was applied. These properties make LTI systems easy to represent and understand graphically.

Linear systems have the property that the output is linearly related to the input. Changing the input in a linear way will change the output in the same linear way. So if the input $\mathbf{x_1(t)}$ produces the output $\mathbf{y_1(t)}$ and the input $\mathbf{x_2(t)}$ produces the output **$y_2(t)$**, then linear combinations of those inputs will produce linear combinations of those outputs. The input **$\{x_1(t)+x_2(t)\}$** will produce the output **$\{y_1(t)+y_2(t)\}$.** Further, the input **$\{a_1x_1(t)+a_2x_2(t)\}$** will produce the output **$\{a_1y_1(t)+a_2y_2(t)\}$** for some constants **a_1** and **a_2**.

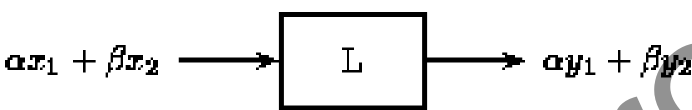
In other words, for a system T over time t, composed of signals **$x_1(t)$** and **$x_2(t)$** with outputs **$y_1(t)$** and **$y_2(t)$** ,

$$T\left[a_1x_1(t)+a_2x_2(t)\right]=a_1T\left[x_1(t)\right]+a_2T\left[x_2(t)\right]=a_1y_1(t)+a_2y_2(t)$$

Homogeneity Principle:



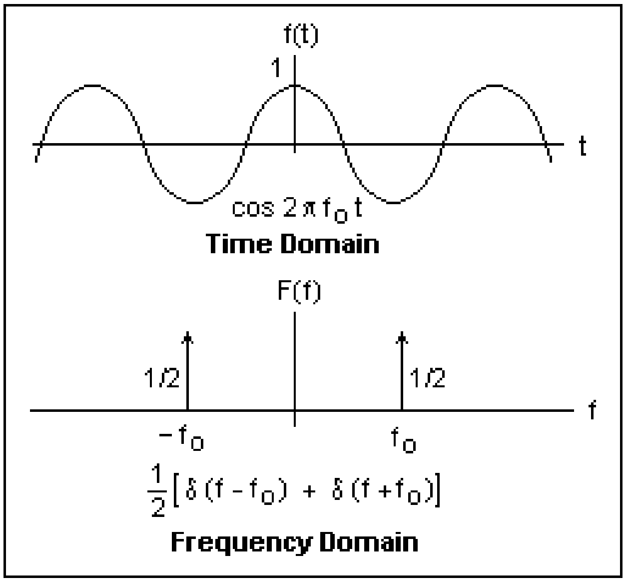
Superposition Principle:



Thus, the entirety of an LTI system can be described by a single function called its **impulse response**. This function exists in the time domain of the system. For an arbitrary input, the output of an LTI system is the convolution of the input signal with the system's impulse response.

Conversely, the LTI system can also be described by its **transfer function**. The transfer function is the Laplace transform of the impulse response. This transformation changes the function from the time domain to the frequency domain. This transformation is important because it turns differential equations into algebraic equations, and turns convolution into multiplication.

In the frequency domain, the output is the product of the transfer function with the transformed input. The shift from time to frequency is illustrated in the following image:



Homogeneity, additivity, and shift-invariance may, at first, sound a bit abstract but they are very useful. To characterize a shift-invariant linear system, we need to measure only one thing: the way the system responds to a unit impulse. This response is called the impulse response function of the system. Once we've measured this function, we can (in principle) predict how the system will respond to any other possible stimulus.

Introduction to Convolution

Because here's not a single answer to define what is? In "Signals and Systems" probably we saw convolution in connection with **Linear Time-Invariant Systems** and the **impulse response for such a system**. This multitude of interpretations and applications is somewhat like the situation with the definite integral.

To pursue the analogy with the integral, in pretty much all applications of the integral there is a general method at work:

- Cut the problem into small pieces where it can be solved approximately.
- Sum up the solution for the pieces, and pass to a limit.

Convolution Theorem

F(g∗f)(s)=Fg(s)Ff(s)

- In other notation: If **f(t)⇔ F(s)** and **g(t) ⇔ G(s) then (g∗f)(t)⇔ G(s)F(s)**

- **In words:** Convolution in the time domain corresponds to multiplication in the frequency domain.

$$(g * f)(t) = \int_0^1 g(t - x) f(x)\, dx$$

- For the Integral to make sense i.e., to be able to evaluate g(t−x) at points outside the interval from 0 to 1, we need to assume that g is periodic. it is not the issue the present case, where we assume that f(t) and g(t) are defined for all t, so the factors in the integral

$$\int_{-\infty}^{\infty} g(t - x) f(x)\, dx$$

Convolution in the Frequency Domain

- In Frequency Domain convolution theorem states that

F(g ∗ f)=Fg ·Ff

- here we have seen that the whole thing is carried out for inverse Fourier transform, as follow:

$F^{-1}(g*f)=F^{-1}g \cdot F^{-1}f$

F(gf)(s)=(Fg∗Ff)(s)

- Multiplication in the time domain corresponds to convolution in the frequency domain.

By applying Duality Formula

F(Ff)(s)=f(−s) or **$F(Ff)=f^{-}$** without the variable.

- To derive the identity **F(gf) = Fg∗Ff**, we assume for convenience, **h = Ff** and **k = Fg**

then we can write as F(gf)=k∗h

- The one thing we know is how to take the Fourier transform of a convolution, so, in the present notation, **F(k∗h)=(Fk)(Fh)**.

But now **$Fk = FFg = g^{-}$**

and likewise **Fh =FFf = f**

So F(k∗h)=g⁻f⁻ =(gf)⁻, or gf =F(k∗h)⁻

Now, finally, take the Fourier transform of both sides of this last equation

FF identity : F(gf)=F(F(k∗h)−)=k∗h =Fg∗Ff

Note: Here we are trying to prove **F(gf)(s) = (Fg∗Ff)(s)** rather than **F(g∗f)=(Ff)(Fg)** Because, it seems more "natural" to multiply signals in the time domain and see what effect this has in the frequency domain, so why not work with **F(fg)** directly**?** But write the integral for **F(gf)**; there's nothing you can do with it to get toward Fg∗Ff.

There is also often a general method of convolutions:

- Usually there's something that has to do with **smoothing** and **averaging**,understood broadly.
- You see this in both the continuous case and the discrete case.

Some of you who have seen convolution in earlier courses,you've probably heard the expression "flip and drag"

Meaning of Flip & Drag: here's the meaning of Flip & Drag is as follow

- Fix a value t.The graph of the function g(x−t) has the same shape as g(x) but shifted to the right by t. Then forming g(t − x) flips the graph (left-right) about the line x = t.
- If the most interesting or important features of g(x) are near x = 0, e.g., if it's sharply peaked there, then those features are shifted to x = t for the function g(t − x) (but there's the extra "flip" to keep in mind).Multiply f(x) and g(t − x) and integrate with respect to x.

Averaging

- I prefer to think of the convolution operation as using one function to smooth and average the other. Say g is used to smooth f in g∗f. In many common applications g(x) is a positive function, concentrated near 0, with total area 1.

$$\int_{-\infty}^{\infty} g(x)\,dx = 1.$$

- Like a sharply peaked Gaussian, for example (stay tuned). Then g(t−x) is concentrated near t and still has area 1. For a fixed t, forming the integral

$$\int_{-\infty}^{\infty} g(t-x)f(x)\,dx$$

- The last expression is like a weighted average of the values of **f(x) near x = t**, weighted by the values of (the flipped and shifted) g. That's the averaging part of the convolution, computing the convolution g∗f at t replaces the value f(t) by a weighted average of the values of f near t.

Smoothing

- Again take the case of an averaging-type function g(t), as above. At a given value of t,**(g ∗ f)(t)** is a weighted average of values of f near t.
- Then Move t a little to a point t_0. Then **(g∗f)(t_0)** is a weighted average of values of f near t_0, which will include values of f that entered into the average near t.
- Thus the values of the convolutions (g∗f)(t) and (g∗f)(t_0) will likely be closer to each other than are the values f(t) and f(t_0). That is, **(g ∗f)(t)** is "smoothing" f as t varies — there's less of a change between values of the convolution than between values of f.

Other identities of Convolution

It's not hard to combine the various rules we have and develop an algebra of convolutions. Such identities can be of great use — it beats calculating integrals. Here's an assortment. (Lower and uppercase letters are Fourier pairs.)

- **(f ·g)∗(h·k)(t) ⇔ (F ∗G)·(H ∗K)(s)**
- **{(f(t)+g(t))·(h(t)+k(t)} ⇔ {[(F + G)∗(H + K)]}(s)**
- **f(t)·(g∗h)(t) ⇔ F ∗(G·H)(s)**

Properties of Convolution

Here we are explaining the properties of convolution in both continuous and discrete domain

- Associative
- Commutative
- Distributive properties

- As a LTI system is completely specified by its impulse response, we look into the conditions on the impulse response for the LTI system to obey properties like memory, stability, invertibility, and causality.
- According to the Convolution theorem in Continuous & Discrete time as follow:

For Discrete system .

$$x[n] \rightarrow \boxed{\text{LTI system}} \rightarrow y[n] = x[n] \otimes h[n]$$

output response of a LTI system to an input x[n]

$$\boxed{y[n] = x[n] * h[n] = \sum_{K=-\infty}^{\infty} x[K]h[n-K]}$$

convolution sum

For Continuous System

$$\boxed{y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

or

$$\boxed{y(t) = x(t) * h(t)}$$

We shall now discuss the important properties of convolution for LTI systems.

1) Commutative property :

- **In Discrete time**: **x[n]*h[n] ⇔ h[n]*x[n]**

Proof: Since we know that y[n] = x[n]*h[n]

$$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

Let us assume n-k = l

so,

$$y[n] = \sum_{l=-\infty}^{+\infty} x[n-l]\,h[l] = h[n] * x[n]$$

- So it clear from the derived expression that ⇒ **x[n]*h[n] ⇔ h[n]*x[n]**
- **In Continuous time:**

Proof

$$\boxed{y(t) = x(t) * h(t) = h(t) * x(t)}$$

or

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

So **x[t]*h[t] ⇔ h[t]*x[t]**

2. Distributive Property

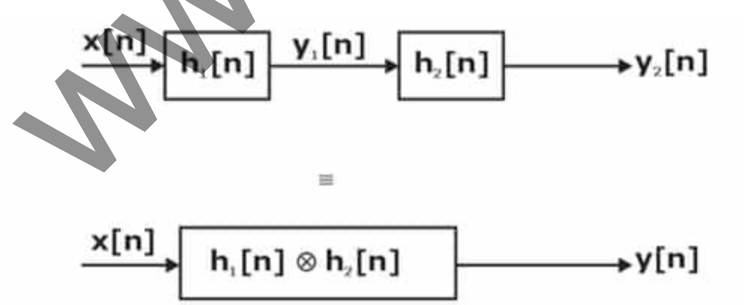
By this Property we will conclude that convolution is distributive over addition.

- **Discrete time:** $x[n]\{\alpha\, h_1[n] + \beta h_2[n]\} = \alpha\,\{x[n]\, h_1[n]\} + \beta\{x[n]\, h_2[n]\}$ α & β are constant.
- **Continuous Time:** $x(t)\{\alpha\, h_1(t) + \beta h_2(t)\} = \alpha\{x(t)h_1(t)\} + \beta\,\{x(t)h_2(t)\}$ α & β are constant.

3. Associative Property

- **Discrete Time** $y[n] = x[n]*h[n]*g[n]$

$x[n] * h_1[n] * h_2[n] = x[n] * (h_1[n] * h_2[n])$

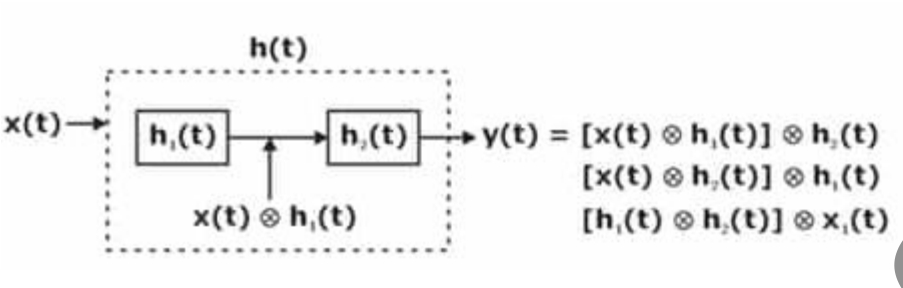


Associative property of convolution sum

- **In Continuous Time:**

$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$

If systems are connected in cascade:



∴ Overall impulse response of the system is:

$$\boxed{h(t) = h_1(t) * h_2(t) * h_3(t) * \ldots\ldots}$$

4. Invertibility

A system is said to be invertible if there exist an inverse system which when connected in series with the original system produces an output identical to input .

$(x*\delta)[n] = x[n]$

$(x*h*h^{-1})[n] = x[n]$

$(h*h^{-1})[n] = (\delta)[n]$

5. Causality

- **Discrete Time**

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

$$h[n] = 0 \qquad \forall n < 0$$

- **Continuous Time**

$$y(t) = \int_{-\infty}^{+\infty} x(v)\, h(t-v)\, dv$$

$$h(t) = 0 \qquad \forall\, t < 0$$

6. Stability

- **Discrete Time**

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty, \text{ in the Discrete domain,}$$

- **Continuous Time**

$$\int_{-\infty}^{\infty} |\, h(t)\, |dt < \infty, \text{ in the Continuous domain.}$$

Laplace Transform

The Laplace Transform is a very important **tool** to analyse any electrical containing by which we can convert the Integral-Differential Equation in Algebraic by converting the given situation in **Time Domain** to **Frequency Domain**

- $$L\{X(t)\} = X(s) = \int_{-\infty}^{\infty} X(t)\,.\,e^{-st}\,dt$$
- is also called bilateral or two-sided Laplace transform.
- If *x*(*t*) is defined for *t*≥0, [i.e., if *x*(*t*) is causal], then

$$L\{x(t)\} = X(s) = \int_0^{\infty} x(t)\,.\,e^{-st}\,dt$$

is also called unilateral or one-sided Laplace transform.

Below we have listed the Following advantage of accepting Laplace transform:

- Analysis of general R-L-C circuits become easier.

- Natural and Forced response can be easily analyzed.
- The circuit can be analyzed with impedances.
- Analysis of stability can be done easiest way.

Statement of Laplace Transform

- The direct Laplace transform or the Laplace integral of a function f(t) defined for $0 \le t < \infty$ is the ordinary calculus integration problem for a given function f(t).
- Its Laplace transform is the function, denoted F(s) = L{f}(s), defined by

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

- A causal signal *x*(*t*) is said to be of exponential order if a real, positive constant σ (where σ is the real part of *s*) exists such that the function, $e^{-\sigma t}|X(t)|$ approaches zero as *t* approaches infinity.
- For a causal signal, if **lim** $e^{-\sigma t}|x(t)|=0$, for $\sigma > \sigma_c$ and if **lim** $e^{-\sigma t}|x(t)|=\infty$ for $\sigma > \sigma_c$ then σ_c is called the abscissa of convergence, (where σ_c is a point on real axis in s-plane).
- The value of s for which the integral

$$\int_{-\infty}^{\infty} x(t)\,.\,e^{-st}\, dt$$

converges is called Region of Convergence (ROC).

- For a causal signal, the ROC includes all points on the s-plane to the right of abscissa of convergence.
- For an anti-causal signal, the ROC includes all points on the s-plane to the left of the abscissa of convergence.
- For a two-sided signal, the ROC includes all points on the s-plane in the region in between two abscissae of convergence.

Properties of the ROC

The region of convergence has the following properties

- ROC consists of strips parallel to the *j*ω-axis in the s-plane.
- ROC does not contain any poles.

- If $x(t)$ is a finite duration signal, $x(t) \neq 0, t_1 < t < t_2$ and is absolutely integrable, the ROC is the entire s-plane.
- If $x(t)$ is a right sided signal, $x(t) = 0, t_1 < t_0$, the ROC is of the form R{s} > max {$R\{p_k\}$}
- If $x(t)$ is a left sided signal $x(t) = 0, t_1 > t_0$, the ROC is of the form R{s} > min {$R\{p_k\}$}
- If $x(t)$ is a double-sided signal, the ROC is of the form $p_1 < R\{s\} < p_2$
- If the ROC includes the jω-axis. Fourier transform exists and the system is stable.

Inverse Laplace Transform

- It is the process of finding $x(t)$ given $X(s)$

$X(t) = L^{-1}\{X(s)\}$

There are two methods to obtain the inverse Laplace transform.

- Inversion using Complex Line Integral

$$x(t) = \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st} ds$$

- Inversion of Laplace Using Standard Laplace Transform Table.

Note A: Derivatives in t → Multiplication by s.

$$\mathcal{L}\{f'(t)\} = \begin{pmatrix} s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F(s) \\ -f(0) \end{pmatrix} = sF(s) - f(0)$$

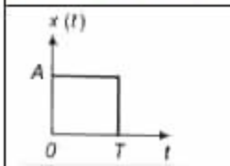
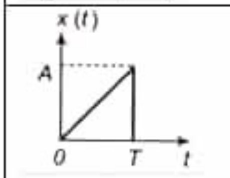
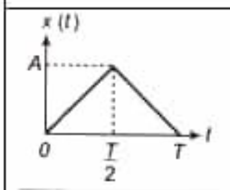
$$\mathcal{L}\{f''(t)\} = \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F(s) \\ -f(0) \\ -f'(0) \end{pmatrix} = s^2F(s) - sf(0) - f'(0)$$

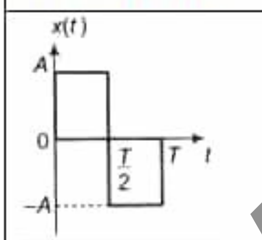
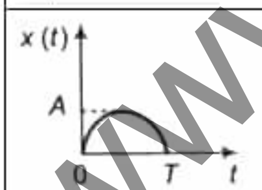
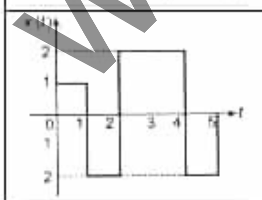
$$\mathcal{L}\{f^{(n)}(t)\} = \begin{pmatrix} s^n \\ s^{n-1} \\ \vdots \\ s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F(s) \\ -f(0) \\ \cdots \\ -f^{(n-2)}(0) \\ -f^{(n-1)}(0) \end{pmatrix}$$
$$= s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

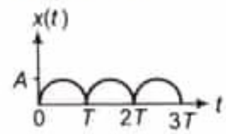
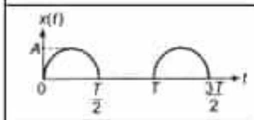
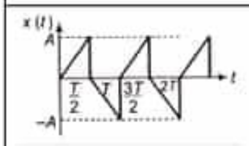
B: Multiplication by t → Derivatives in s.

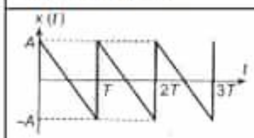
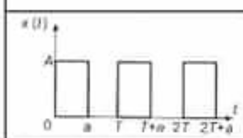
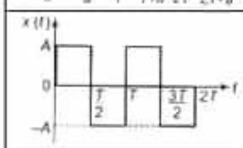
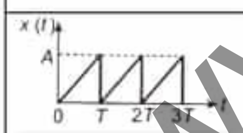
$$\mathcal{L}\{tf(t)\} = -F'(s)$$
$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Laplace Transform of Some Standard Signals

Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t) = A;\ 0 < t < T$ $= 0;\ t > T$	$X(t) = \frac{A}{s}(1 - e^{-sT})$
	$x(t) = \frac{At}{T}$ $0 < t < T$ $= 0;\ t > T$	$X(t) = \frac{A}{Ts^2}$ $[1 - e^{-sT}(1 + sT)]$
	$x(t) = \frac{At}{T}; 0 < t < \frac{T}{2}$ $= 2A - \frac{2At}{T}$ $= 0;\ t > T$	$X(t) = \frac{A}{Ts^2}\left(1 - e^{\frac{-sT}{2}}\right)^2$

Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t) = A; 0 < t < \frac{T}{2}$ $= -A; \frac{T}{2} < t < T$	$X(t) = \frac{A}{s}\left(1 - e^{\frac{-sT}{2}}\right)^2$
	$x(t) = A \sin t;$ $0 < t < T$ $= 0;\ T > T$	$X(t) = \frac{A}{s^2+1}\left(e^{-sT} + 1\right)$
	$x(t) = 1; 0 < t < 1$ $= -2; 1 < t < 2$ $= -2; 4 < t < 5$ $= 0; t > 5$	$X(s) = \frac{1}{s}(1 - 3e^{-s} +$ $4e^{-2s} - 4e^{-4s} + 2e^{-2s})$

Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t)= A\sin t;$ $0< t< T$ and $x(t+nT)= x(t)$	$X(t) = \frac{A}{(s^2+1)}\left(\frac{1-e^{-sT}}{1-e^{-sT}}\right)$
	$x(t)\cong A\sin t;$ $0< t< \frac{T}{2}$ $= 0; \frac{T}{2}< t< T$ and $(t+nT)= x(t)$	$X(t) = \frac{A}{(s^2+1)\left(1-e^{\frac{-sT}{2}}\right)}$
	$x(t)= \frac{2At}{T}; 0< t< \frac{T}{2}$ $= A-\frac{2At}{T}; \frac{T}{2}< t< T$ and $x(t+nT)= x(t)$	$X(s) = \frac{2A\left[1-\left(1+\frac{Ts}{2}\right)e^{-\frac{sT}{2}}\right]}{Ts^2\left(1-e^{-\frac{sT}{2}}\right)}$

Waveform	$x(t)$	$x(s) = L\{x(t)\}$
	$x(t)= A\frac{2At}{T};$ $0< t< T$ and $x(t+nT)= x(t)$	$X(s) = \frac{2A}{Ts}\left(\frac{T}{2}\frac{1+e^{-sT}}{1-e^{-sT}}-\frac{1}{s}\right)$
	$x(t)= A; 0< t< a$ $= 0; a< t< T$ and $(t+nT)= x(t)$	$X(s) = \frac{A}{s}\left(\frac{1-e^{-as}}{1-e^{-sT}}\right)$
	$x(t)= A; 0< t< \frac{T}{2}$ $= A; \frac{T}{2}< t< T$ and $(t+nT)= x(t)$	$X(t) = \frac{A}{s}\left(\frac{1-e^{-\frac{sT}{2}}}{1+e^{-\frac{sT}{2}}}\right)$
	$x(t)= \frac{At}{T}; 0< t< T$ and $(t+nT)= x(t)$	$X(s) = \frac{A}{Ts^2}$ $\left[\frac{1-e^{-sT}(1+sT)}{1+e^{sT}}\right]$

Some Standard Laplace Transform Pairs

$x(t)$	$x(s)$	**ROC**
$\delta(t)$	1	Entire s-plane
$u(t)$	1/s	σ > 0
$tu(t)$	$1/s^2$	σ > 0
$\frac{t^{n-1}}{(n-1)!}u(t)$, where $n = 1, 2, 3$	$1/s^n$	σ > 0
$e^{-at}u(t)$	$\frac{1}{s+a}$	σ > -a

$x(t)$	$x(s)$	**ROC**
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	σ < -a
$t^n u(t)$ where, $n = 1, 2, 3$	$\frac{n!}{s^{n+1}}$	σ > 0
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	σ > -a
$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t)$ where, $n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$	σ > -a

$x(t)$	$x(s)$	**ROC**
$t^n e^{-at} u(t)$ where, $n = 1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$	$\sigma > -a$
$\sin \omega_0 t\, u(t)$	$\frac{\omega_0}{(s^2+\omega_0^2)}$	$\sigma > 0$
$\cos \omega_0 t\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\sigma > 0$
$\sin h\, \omega_0 t\, u(t)$	$\frac{\omega_0}{s^2-\omega_0^2}$	$\sigma > \omega_0$

$x(t)$	$x(s)$	**ROC**
$\cos h\, \omega_0 t\, u(t)$	$\frac{s}{s^2-\omega_0^2}$	$\sigma > \omega_0$
$e^{-at} \sin \omega_0 t\, u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\sigma > -a$
$e^{-at} \cos \omega_0 t\, u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\sigma > -a$

Properties of Laplace Transform

Property	Time Domain Signal	s-domain Signal
Amplitude scaling	$A\,x(t)$	$A\,X(s)$
Linearity	$a_1x_1(t) \pm a_2x_2(t)$	$a_1X_1(s) \pm a_2X_2(s)$
Time differentiation	$\frac{d}{dt}x(t)$	$sX(s) - x(0)$
	$\frac{d^n}{dt^n}x(t)$, where $n = 1, 2, 3, \ldots$	$s^n X(s) - \sum_{k=1}^{n} s^{n-k} \left. \frac{d^{(k-1)}x(t)}{dt^{k-1}} \right\|_{t=0}$

Property	Time Domain Signal	s-domain Signal
Time integration	$\int x(t)\,dt$	$\frac{X(s)}{s} + \frac{[\int x(t)dt]_{t=0}}{s}$
	$\int \ldots \pm \int x(t)\,(dt)^n$, where $n = 1,2,3,\ldots$	$\frac{X(s)}{s} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}} [\int \ldots \int x(t)dt)^k]]_{t=0}$
Frequency shifting	$e^{\pm at}x(t)$	$X(s \pm a)$
Time shifting	$x(t \pm \alpha)$	$e^{\pm as}X(s)$
Frequency differentiation	$t\,x(t)$	$-\frac{dX(s)}{ds}$
	$t^n x(t)$, where $n = 1,2,3,\ldots$	$(-1)^n \frac{d^n}{ds^n}X(s)$

Property	Time Domain Signal	s-domain Signal
Frequency integration	$\frac{1}{t}x(t)$	$\int_s^{\infty} X(s)\,ds$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$
Periodicity	$x(t+nT)$	$\frac{1}{1-e^{-sT}}\int_0^T x_1(t)e^{-st}dt$ where, $x_1(t)$ is one period of $x(t)$
Initial value Theorem	$\lim_{t\to 0} x(t) = x(0)$	$\lim_{s\to\infty} s\,X(s)$
Final value theorem	$\lim_{t\to\infty} x(t) = x(\infty)$	$\lim_{s\to 0} s\,X(s)$
Convolution theorem	$x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)\,x_2(t-\lambda)d\lambda$	$X_1(s)X_2(s)$

Key Points

- The convolution theorem of Laplace transform says that Laplace transform of convolution of two time-domain signals is given by the product of the Laplace transform of the individual signals.
- The zeros and poles are two critical complex frequencies at which a rational function of a takes two extreme value zero and infinity respectively.

Fourier Series & Fourier Transform

Fourier Theorem

Any arbitrary continuous-time signal x(t), which is periodic with a fundamental period T_0, can be expressed as a series of harmonically related sinusoids whose frequencies are multiples of fundamental frequency or first harmonic. In other words, any periodic function of (t) can be represented by an infinite series of sinusoids called the **Fourier Series**.

The periodic waveform is expressed in the form of **Fourier series**, while a non-periodic waveform may be expressed by the **Fourier transform**.

The different forms of the Fourier series are given as follows.

(i) Trigonometric Fourier series

(ii) Complex exponential Fourier series

(iii) Polar or harmonic form Fourier series.

Trigonometric Fourier Series

Any arbitrary periodic function x(t) with fundamental period T_0 can be expressed as follows.

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t} \quad .....(i)$$

This is called the trigonometric Fourier series representation of signal x(t). Here, $\omega_0 = 2\pi/T_0$ is the fundamental frequency of x(t), and coefficients a_0, a_n, and b_n are referred to as the trigonometric continuous-time Fourier series (CTFS) coefficients. The coefficients are calculated as follows.

Fourier Series Coefficient

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt \quad .....(ii)$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt \quad .....(iii)$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt \quad ....(iv)$$

From equation (ii), it is clear that coefficient a_0 represents the average or mean value (also referred to as the dc component) of signal x(t).

In these formulas, the limits of integration are either ($-T_0/2$ to $+T_0/2$) or (0 to T_0). In general, the limit of integration is any period of the signal, and so the limits can be from (t_1 to $t_2 + T_0$), where t_1 is any time instant.

Trigonometric Fourier Series Coefficients for Symmetrical Signals

If the periodic signal x(t) possesses some symmetry, then the continuous-time Fourier series (CTFS) coefficients become easy to obtain. The various types of symmetry and simplification of Fourier series coefficients are disused below.

Consider the Fourier series representation of a periodic signal x(t) defined in the equation.

$$\text{(i)}\ x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

where
$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt$$

Even Symmetry: x(t) = x(−t)

If x(t) is an even function, then product x(t) $\sin\omega_0 t$ is odd, and integration in equation (iv) becomes zero. That is $b_n = 0$ for all n, and the Fourier series representation expressed as

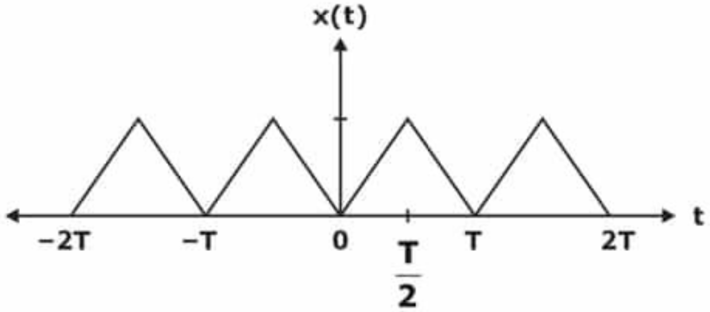
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

where,
$$a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\,dt$$

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt$$

For example, the signal x(t) shown below figure has even symmetry, so $b_n = 0$, and the Fourier series expansion of x(t) is given as

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\left[\cos\omega_0 t + \frac{\cos 3\omega_0 t}{(3)^2} + \frac{\cos 5\omega_0 t}{(5)^2} + \ldots..\right]$$



Waveform with even symmetry

The trigonometric Fourier series representation of even signals contains cosine terms only. The constant a_0 may or may not be zero.

Odd Symmetry: x(t) = −x(−t)

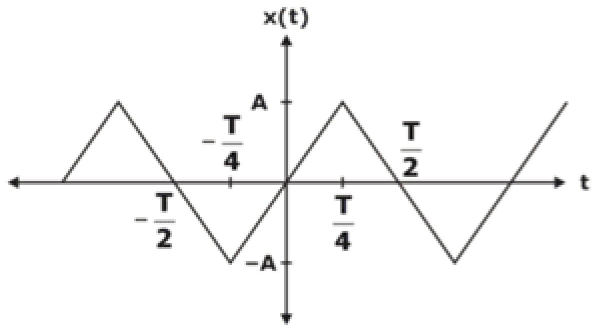
If x(t) is an odd function, then product x(t) $\cos\omega_0 t$ is also odd and integration in equation (iii) becomes zero i.e. $a_n = 0$ for all n. Also, $a_0 = 0$ because an odd symmetric function has a zero-average value. The Fourier series representation is expressed as

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\text{where,} \qquad b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin n\omega_0 t\,dt$$

For example, the signal x(t) shown in below figure is odd symmetric, so $a_n = a_0 = 0$, and the Fourier series expansion of x(t) is given as

$$x(t) = \frac{8A}{\pi^2}\left[\sin\omega_0 t - \frac{\sin 3\omega_0 t}{(3)^2} + \frac{\sin 5\omega_0 t}{(5)^2} - \frac{\sin 7\omega_0 t}{(7)^2} + \ldots\right]$$



Waveform with odd symmetry

Fourier Sine Series

The Fourier Sine series can be written as

$$S(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{n=1}^{\infty} b_n \sin nx$$

------(2)

- Sum S(x) will inherit all three properties:
- **(i): Periodic S(x +2π)=S(x); (ii): Odd S(−x)=−S(x); (iii): S(0) = S(π)=0**
- Our first step is to compute from S(x) the number b_k that multiplies sin(kx).

Suppose $S(x)=\sum b_n \sin(nx)$. Multiply both sides by sin(kx). Integrate from 0 to π in Sine Series in equation (2)

$$\int_0^{\pi} S(x)\sin kx\, dx = \int_0^{\pi} b_1 \sin x \sin kx\, dx + \cdots + \int_0^{\pi} b_k \sin kx\ \sin kx\, dx + \cdots$$

- On the right side, all integrals are zero except for n = k. Here the property of “orthogonality” will dominate. The sines make 90° angles in function space when their inner products are integrals from 0 to π.
- Orthogonality for sine Series

Condition for Orthogonality:

$$\int_0^{\pi} \sin nx \ \sin kx \, dx = 0 \quad \text{if} \quad n \neq k$$

------(3)

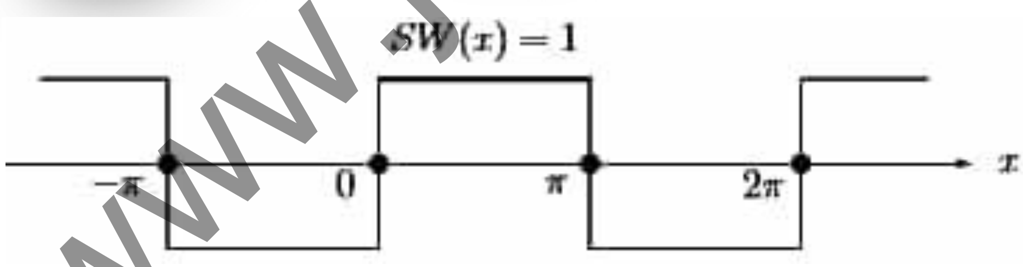
- Zero comes quickly if we integrate the term cos(mx) from 0 to π. ⇒ $_0\int^{\pi}$ cos(mx) dx = 0-0=0.
- Integrating cos(mx) with m = n−k and m = n + k proves the orthogonality of the sines.
- The exception is when n = k. Then we are integrating $\sin^2(kx) = 1/2 - 1/2 \cos(2kx)$

$$\int_0^{\pi} \sin kx \ \sin kx \, dx = \int_0^{\pi} \frac{1}{2} \, dx - \int_0^{\pi} \frac{1}{2} \cos 2kx \, dx = \frac{\pi}{2}.$$

$$b_k = \frac{2}{\pi} \int_0^{\pi} S(x) \sin kx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} S(x) \sin kx \, dx.$$

------(4)

- Notice that S(x)sin(kx is even (equal integrals from −π to 0 and from 0 to π).
- We will immediately consider the most important example of a Fourier sine series. S(x) is an odd square wave with SW(x) = 1 for 0<x<π. It is an odd function with period 2 π, that vanishes at x=0 and x= π.



Example:

As given above, finding the Fourier sine coefficients bk of the square wave SW(x).

Solution:

For k =1,2,...using the formula of sine coefficient with S(x)=1 between 0 and π:

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin kx \, dx = \frac{2}{\pi}\left[\frac{-\cos kx}{k}\right]_0^{\pi} = \frac{2}{\pi}\left\{\frac{2}{1}, \frac{0}{2}, \frac{2}{3}, \frac{0}{4}, \frac{2}{5}, \frac{0}{6}, \cdots\right\}$$

- Then even-numbered coefficients b_{2k} are all zero because $\cos(2k\pi) = \cos(0) = 1$.
- The odd-numbered coefficients $b_k = 4/\pi k$ decrease at the rate $1/k$.
- We will see that same $1/k$ decay rate for all functions formed from smooth pieces and jumps. Put those coefficients $4/\pi k$ and zero into the Fourier sine series for SW(x).

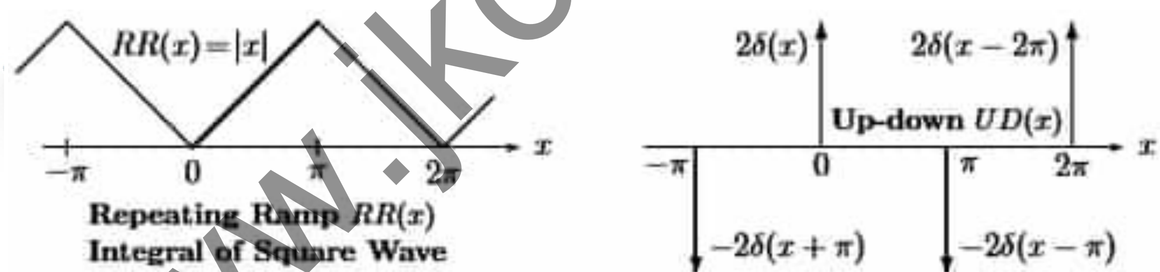
$$\textbf{Square wave} \qquad SW(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right]$$

Fourier Cosine Series

The cosine series applies to even functions with $C(-x)=C(x)$ as

$$C(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

-----(5)



Cosine has period 2π shown as above in figure two even functions, the repeating ramp RR(x), and the up-down train UD(x) of delta functions.

- That sawtooth ramp RR is the integral of the square wave. The delta functions in UD give the derivative of the square wave. RR and UD will be valuable examples, one smoother than SW and one less smooth.
- First, we find formulas for the cosine coefficients a_0 and a_k. The constant term a_0 is the average value of the function C(x):

$$a_0 = \textbf{Average} \qquad a_0 = \frac{1}{\pi}\int_0^{\pi} C(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(x)\,dx.$$

-----(6)

- We will integrate the cosine series from 0 to π. On the right side, the integral of **a_0=a_0π** (divide both sides by π). All other integrals are zero.

$$\int_0^{\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_0^{\pi} = 0 - 0 = 0.$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} C(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} C(x)\cos kx\,dx$$

- Again the integral over a full period from −π to π (also 0 to 2π) is just doubled.

Orthogonality Relations of Fourier Series

Since from the Fourier Series Representation, we concluded that a periodic Signal it could be written as

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\frac{\pi}{L}t)\,dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\frac{\pi}{L}t)\,dt$$

-------(7)

The condition of orthogonality is as follows:

$$\frac{1}{L}\int_{-L}^{L}\cos\left(n\frac{\pi}{L}t\right)\cos\left(m\frac{\pi}{L}t\right)dt=\begin{cases}1 & n=m\neq 0\\ 0 & n\neq m\\ 2 & n=m=0\end{cases}$$

$$\frac{1}{L}\int_{-L}^{L}\cos\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)dt=0$$

$$\frac{1}{L}\int_{-L}^{L}\sin\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)dt=\begin{cases}1 & n=m\neq 0\\ 0 & n\neq m\end{cases}$$

Proof of the orthogonality relations:

This is just a straightforward calculation using the periodicity of sine and cosine and either (or both) of these two methods:

$$\text{Method 1: use } \cos at=\frac{e^{iat}+e^{-iat}}{2}, \text{ and } \sin at=\frac{e^{iat}-e^{-iat}}{2i}$$

$$\text{Method 2: Use the trig identity } \cos(\alpha)\cos(\beta)=\frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)),$$
$$\text{and the similar trig identity for } \cos(\alpha)\sin(\beta) \text{ and } \sin(\alpha)\sin(\beta)$$

Energy in Function = Energy in Coefficients

There is also another important equation (**the energy identity**) that comes from integrating $(F(x))^2$. When we square the Fourier series of F(x) and integrate from −π to π, all the “cross-terms” drop out. The only nonzero integrals come from 1^2 and cos2 kx and sin2 kx, multiplied by a_0^2,a_k^2 b_k^2.

- Energy in F(x) equals the energy in the coefficients.
- The left-hand side is like the length squared of a vector, except the vector is a function.
- The right-hand side comes from an infinitely long vector of a’s and b’s.
- If the lengths are equal, which says that the Fourier transforms from function to vector is like an orthogonal matrix.
- Normalized by constants √2π and √π, we have an orthonormal basis in function space.

Complex Fourier Series

- In place of separate formulas for $\mathbf{a_0}$ and $\mathbf{a_k}$ and $\mathbf{b_k}$, we may consider one formula for all the complex coefficients c_k.
- So that the function F(x) will be complex, The Discrete Fourier Transform will be much simpler when we use N complex exponentials for a vector.

The exponential form of the Fourier series of a periodic signal x(t) with period T_0 is defined as

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$

where ω_0 is the fundamental frequency given as $\omega_0 = 2\pi / T_0$. The exponential Fourier series coefficients c_n are calculated from the following expression

$$C_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

- Since $c_0 = a_0$ is still the average of F(x), because $e_0 = 1$.
- The orthogonality of e^{inx} and e^{ikx} is to be checked by integrating.

Orthogonality of e^{inx} and e^{ikx}

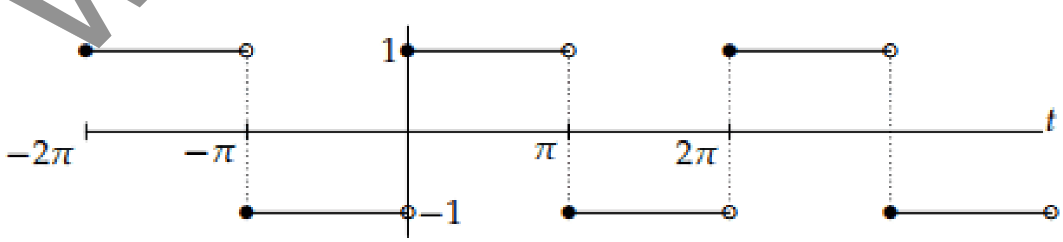
$$\int_{-\pi}^{\pi} e^{i(n-k)x} dx = \left[\frac{e^{i(n-k)x}}{i(n-k)}\right]_{-\pi}^{\pi} = 0.$$

Example:

Compute the Fourier series of f(t), where f(t) is the square wave with period 2π. defined over one period.

$$f(t) = \begin{cases} -1 & \text{for } -\pi \le t < 0 \\ 1 & \text{for } 0 \le t < \pi \end{cases}$$

The graph over several periods is shown below.



Solution:

Computing a Fourier series means computing its Fourier coefficients. We do this using the integral formulas for the coefficients given with Fourier's theorem in the previous note. For convenience, we repeat the theorem here.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

By applying these formulas to the above waveform, we have to split the integrals into two pieces corresponding to where f(t) is +1 and where it is −1.

thus for $n \neq 0$;

$$a_n = -\frac{\sin(nt)}{n\pi}\Big|_{-\pi}^{0} + \frac{\sin(nt)}{n\pi}\Big|_{0}^{\pi} = 0$$

for $n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0.$$

Likewise

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)\,dt \\ &= \frac{\cos(nt)}{n\pi}\Big|_{-\pi}^{0} - \frac{\cos(nt)}{n\pi}\Big|_{0}^{\pi} = \frac{1-\cos(-n\pi)}{n\pi} - \frac{\cos(n\pi)-1}{n\pi} \\ &= \frac{2}{n\pi}(1-\cos(n\pi)) = \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}. \end{aligned}$$

We have used the simplification $\cos n\pi = (-1)^n$ to get a nice formula for the coefficients b_n.

This then gives the Fourier series for f(t)

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right)$$

Fourier Transform:

Fourier transform is a transformation technique that transforms non-periodic signals from the continuous-time domain to the corresponding frequency domain. The Fourier transform of a continuous-time non-periodic signal x(t) is defined as

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

where X(jω) is the frequency domain representation of the signal x(t), and F denotes the Fourier transformation. The variable 'ω' is the radian frequency in rad/sec. Sometimes X(jω) is also written as X(ω).

If the frequency is represented in terms of cyclic frequency *f* (in Hz), then the above equation is written as

$$X(jf) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Note:

The signal x(t) and its Fourier transform X(jω) are said to form a Fourier transform pair denoted as

$$x(t) \xleftrightarrow{F} X(j\omega)$$

Existence of Fourier Transform:

A function x(t) has a unique Fourier transform if the following conditions are satisfied, which are also referred to as Dirichlet Conditions:

Dirichlet Conditions:

(i) is absolutely integrable. That is,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

(ii) x(t) has a finite number of maxima and minima and a finite number of discontinuities within any finite interval.

The above conditions are only sufficient conditions but not necessary for the signal to be Fourier transformable. For example, the signals u(t),r(t), and cos (ω_0t) are not absolutely integrable but still possess a Fourier transform.

Magnitude and Phase Spectrum:

The Fourier transform X(jω) of a signal x(t) is, in general, the complex form that can be expressed as

$$X(j\omega) = |X(j\omega)| \underline{|X(j\omega)}$$

The plot of |X(jω)| versus ω is called the magnitude spectrum of *x(t),* and the plot of $\underline{|X(j\omega)}$ versus ω is called the phase spectrum. The amplitude (magnitude) and phase spectra are together called Fourier spectrum, which is nothing but the frequency response of X(jω) for the frequency range

$$-\infty < \omega < \infty$$

Inverse Fourier Transform:

The inverse Fourier transform of X(jω) is given as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

This method of calculating the inverse Fourier transform seems difficult as is involves integration. There is another method to obtain inverse Fourier transform using partial fraction. Let a rational Fourier transform is given as

$$X(j\omega) = \frac{N(j\omega)}{D(j\omega)}$$

X(jω) can be expressed as a ratio of two factorized polynomials in jω as shown below.

$$X(j\omega) = \frac{(j\omega + z_1)(j\omega + z_2)(j\omega + z_3)....}{(j\omega + p_1)(j\omega + p_2)(j\omega + p_3)....}$$

By partial fraction expansion technique, the above can be expressed as shown below.

$$X(j\omega) = \frac{k_1}{j\omega + p_1} + \frac{k_2}{j\omega + p_2} + \frac{k_3}{j\omega + p_3}.$$

where k_1 ,k_2 ......k_n calculated depending on whether the roots are real and simple or repeater or complex.

Properties of Fourier Transform:

There are some properties of continuous-time Fourier transform (CTFT) based on the transformation of signals, which are listed below.

a. Linearity:

The linearity property states that the linear combination of signals in the time domain is equivalent to a linear combination of their Fourier transform in the frequency domain.

$$x_1(t) \xleftrightarrow{F} X_1(j\omega)$$

$$x_2(t) \xleftrightarrow{F} X_2(j\omega)$$

$$ax_1(t) + bx_2(t) \xleftrightarrow{F} aX_1(j\omega) + bX_2(j\omega)$$

where a and b are any arbitrary constants.

b. Time Shifting:

The time-shifting property states that the delay of t_0 in the time domain is equivalent to multiplication of $e^{-j\omega t_0}$ with its Fourier transform. It implies that the amplitude spectrum of the original signal does not change, but the phase spectrum is modified by a factor of $-j\omega t_0$.

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(t - t_0) \longleftrightarrow X(j\omega)e^{-j\omega t_0}$$

c. Conjugation and Conjugate Symmetry:

If, $x(t) \xleftrightarrow{F} X(j\omega)$, then $x^*(t) \xleftrightarrow{F} X^*(-j\omega)$

If $x(t)$ is real, then $X(-j\omega) = X^*(j\omega)$

d. Time Scaling

Time scaling property states that the time compression of a signal in the time domain is equivalent to expansion in the Frequency domain and vice-versa,

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right), \qquad a \neq 0$$

e. Differentiation in Time-Domain

The time differentiation property states that differentiation in the time domain is equivalent to the multiplication of jω in the frequency domain.

if, $x(t) \xleftrightarrow{F} X(j\omega)$

then, $\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$

provided the derivative $dx(t)/dt$ exists at all time t.

In general, $\frac{d^n x(t)}{dt^n} \xleftrightarrow{F} (j\omega)^n X(j\omega)$

f. Integration in Time-Domain:

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

g. Differentiation in Frequency Domain:

The differentiation of Fourier transform in the frequency domain is equivalent to the multiplication of time-domain signal with -jt .

Differentiation in Frequency Domain

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$tx(t) \xleftrightarrow{F} j\frac{dX(j\omega)}{d\omega}$$

$$t^n x(t) \xleftrightarrow{F} (j)^n \frac{dX(j\omega)}{d\omega}$$

h. Frequency Shifting:

The frequency-shifting property states that a shift of ω_0 in frequency is equivalent to multiplying the time domain signal by $e^{j\omega_0 t}$

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{F} X\left[j(\omega - \omega_0)\right]$$

i. Duality Property:

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$X(t) \xleftrightarrow{F} 2\pi x(-j\omega)$$

j. Time Convolution:

Convolution between two signals in the time domain is equivalent to the multiplication of Fourier transforms of the two signals in the frequency domain.

$$x_1(t) \xleftrightarrow{F} x_1(j\omega)$$

$$x_2(t) \xleftrightarrow{F} x_2(j\omega)$$

$$x_1(t) * x_2(t) \xleftrightarrow{F} X_1(j\omega) X_2(j\omega)$$

k. Frequency Convolution:

Convolution in the frequency domain (with a normalization factor of 2π) is equivalent to multiplying the signals in the time domain.

$$x_1(t) \xleftrightarrow{F} X_1(j\omega)$$

$$x_2(t) \xleftrightarrow{F} X_2(j\omega)$$

$$x_1(t)x_2(t) \xleftrightarrow{F} \frac{1}{2\pi}\left[X_1(j\omega) * X_2(j\omega)\right]$$

l. Area Under x(t):

If X(jω) is the Fourier transform of x(t), then,

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$

that is, the area under a time function x(t) is equal to the value of its Fourier transform evaluated at ω= 0

m. Area Under X(jω):

If X(jω) is the Fourier transform of x(t), then,

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,d\omega$$

$$\int_{-\infty}^{\infty} X(j\omega)\,d\omega = 2\pi x(0)$$

n. Parseval's Energy Theorem:

If X(jω) is the Fourier transform of an energy signal x(t). then

$$E_x = \int_{-\infty}^{\infty} \left|x(t)^2\right| dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X(j\omega)\right|^2 d\omega$$

where E_x is the total energy of the signal x(t).

Z Transform & Sampling Theorem

Sampling Theorem

The sampling process is usually described in the time domain. In this process, an analog signal is converted into a corresponding sequence of samples that are usually spaced uniformly in time. Consider an arbitrary signal x(t) of finite energy, which is specified for all time as shown in figure 1(a).

Suppose that we sample the signal x(t) instantaneously and at a uniform rate, once every T_S second, as shown in figure 1(b). Consequently, we obtain an infinite sequence of samples spaced T_S seconds apart and denoted by {x(NTS)}, where n takes on all possible integer values.

Thus, we define the following terms:

1. **Sampling Period:** The time interval between two consecutive samples is referred to as the sampling period. In figure 1(b), T_S is the sampling period.
2. **Sampling Rate:** The reciprocal of the sampling period is referred to as sampling rate, i.e.

$f_S = 1/T_S$

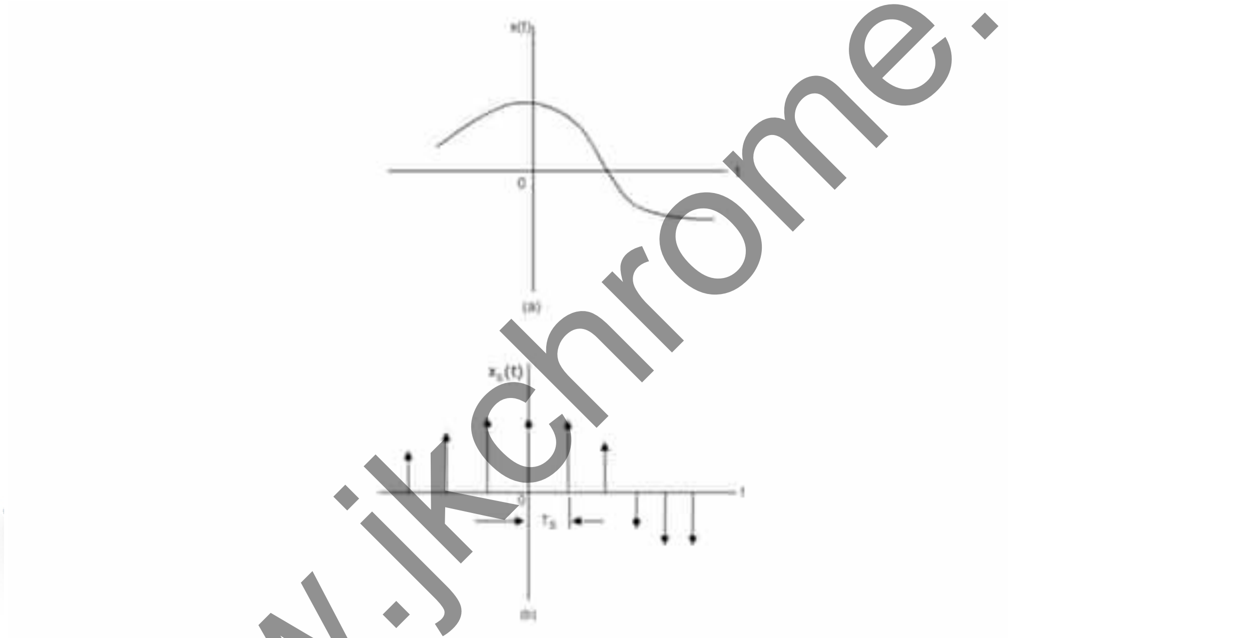


Figure 1: Illustration of Sampling Process: (a) Message Signal, (b) Sampled Signal

Sampling theorem provides both a method of reconstruction of the original signal from the sampled values and also gives a precise upper bound on the sampling interval required for distortion less reconstruction. It states that

- A band-limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signal at instants of time separated by 1/2W seconds.
- A band-limited signal of the finite energy, which has no frequency components higher than W Hertz, may be completely recovered from a knowledge of its samples taken at the rate of 2W samples per second.

Aliasing & Anti-aliasing

- **Aliasing** is such an effect of violating the Nyquist-Shannon sampling theory. During sampling the baseband spectrum of the sampled signal is mirrored to every multifold of the sampling frequency. These mirrored spectra are called alias.
- The easiest way to prevent aliasing is the application of a steep-sloped low-pass filter with half the sampling frequency before the conversion. Aliasing can be avoided by keeping $F_s > 2F_{max}$.
- Since the sampling rate for an analog signal must be at least two times as high as the highest frequency in the analog signal in order to avoid aliasing. So in order to avoid this, the analogue signal is then filtered by a low pass filter prior to being sampled, and this filter is called an anti-aliasing filter. Sometimes the reconstruction filter after a digital-to-analogue converter is also called an anti-aliasing filter.

Explanation of Sampling Theorem

Consider a message signal m(t) bandlimited to W, i.e.

$M(f) = 0 \qquad \text{For } |f| \geq W$

Then, the sampling frequency f_S, required to reconstruct the bandlimited waveform without any error, is given by

$F_s \geq 2\,W$

Nyquist Rate

Nyquist rate is defined as the minimum sampling frequency allowed to reconstruct a bandlimited waveform without error, i.e.

$f_N = \min\{f_S\} = 2W$

Where W is the message signal bandwidth, and f_S is the sampling frequency.

Nyquist Interval

The reciprocal of Nyquist rate is called the Nyquist interval (measured in seconds), i.e.

$$T_N = \frac{1}{f_N} = \frac{1}{2W}$$

Where f_N is the Nyquist rate, and W is the message signal bandwidth.

The Z - Transform of a discrete-time signal x[n] is defined as

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n].z^{-n}$$

where z = r.e$^{j\omega}$

- The discrete-time Fourier Transform (DTFT) is obtained by evaluating Z-Transform at z = **e$^{j\omega}$**
- The z-transform defined above has both sided summation. It is called bilateral or both sided Z-transform.

Unilateral (one-sided) z-transform

- The unilateral z-transform of a sequence x[n] is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Region of Convergence (ROC):

- ROC is the region where z-transform converges. It is clear that z-transform is an infinite power series. The series is not convergent for all values of z.

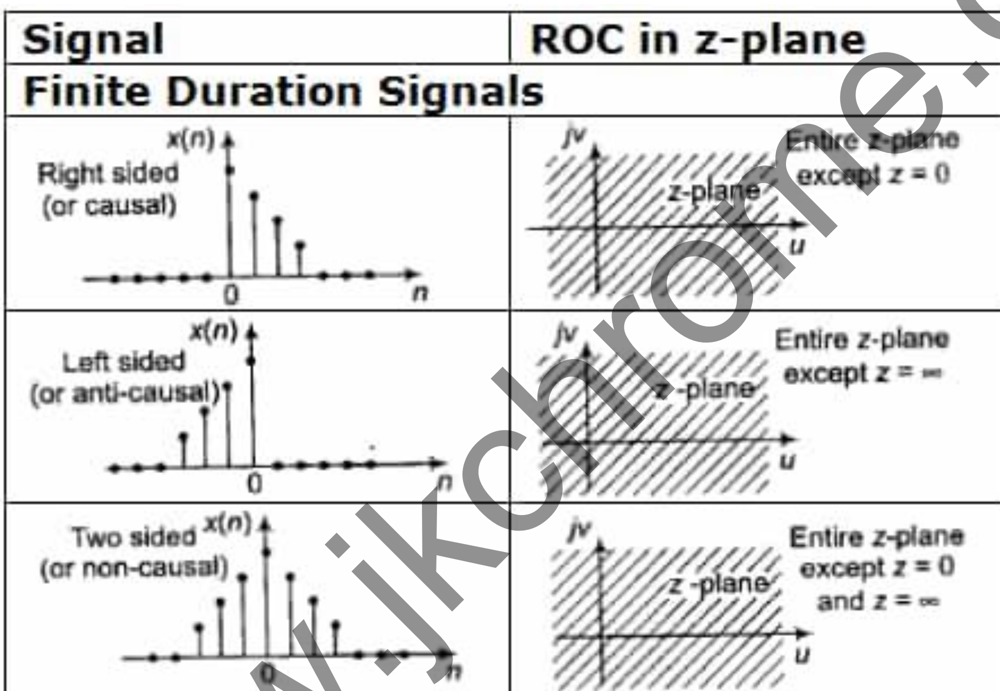
Significance of ROC

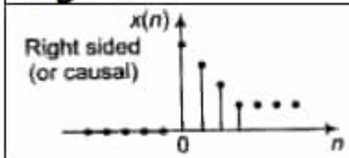
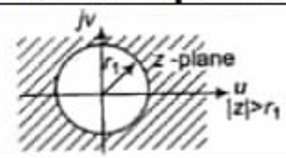
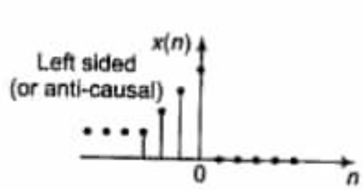
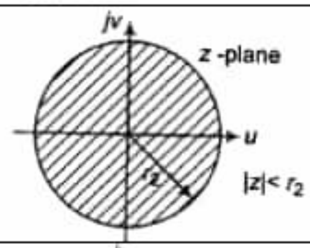
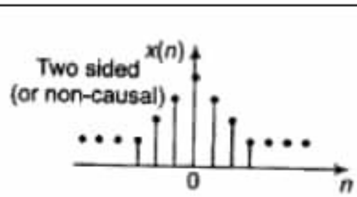
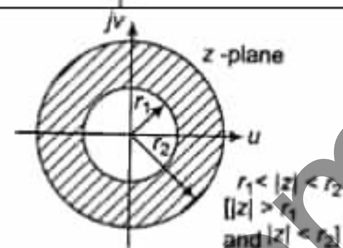
- ROC gives an idea about values of z for which z-transform can be calculated.
- ROC can be used to determine the causality of the system.
- ROC can be used to determine the stability of the system.

Summary of ROC of Discrete-Time Signals for the sequences

Sequence	ROC
Finite, right sided (causal)	Entire z-plane except z = 0
Finite, left sided (anti-causal)	Entire z- plane except z = ∞
Finite, two sided (non-causal)	Entire z-plane except z = 0 and z = ∞
Infinite, right sided (causal)	Exterior of circle of radius r_1, where $\lvert z\rvert > r_1$
Infinite, left sided (anti-causal)	Interior of circle of radius r_2, where $\lvert z\rvert < r_2$
Infinite, two sided (non-causal)	The area between two circles of radius r_2 and r where, $r_2 > r_1$ and $r_1 < \lvert z\rvert\ r_2$, (i. e., $\lvert z\rvert > r_1$ and $\lvert z\rvert < r_2$)

Characteristic Families of Signals and Corresponding ROC



Signal	ROC in z-plane
Infinite Duration Signals	
Signal	**ROC in z-plane**
	
	
	

Note: X(z) = z{x(n)} ; X_1 (z) = Z {x_l (n)} ; X_2(z) = z{x_2 (n)}; Y(z) =z (y (n))

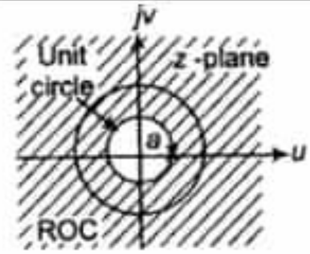
Summary of Properties of z- Transform:

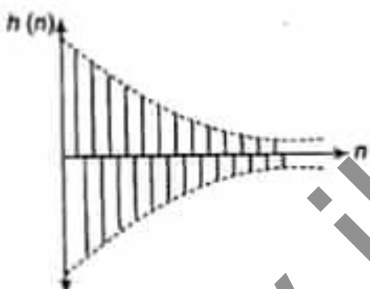
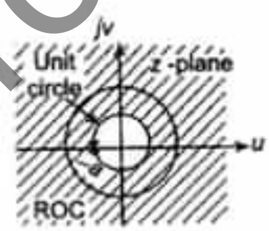
Property		**Discrete Time Signal**	**z-Transform**
Linearity		$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$
Shifting $(m \geq 0)$	x(n) for $n \geq 0$	$x(n-m)$ $x(n+m)$	$z^{-m}X(z)+\sum_{i=1}^{m}$ $x(-i)z^{-(m-i)}$ $z^{m}X(z)=\sum_{i=0}^{m-1}$ $x(i)z^{m-i}$
	x(n) for all n	$x(n-m)$ $x(n+m)$	$z^{-m}X(z)$ $z^{m}X(z)$

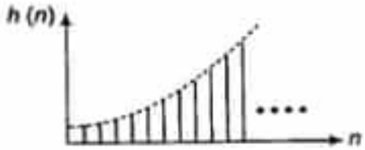
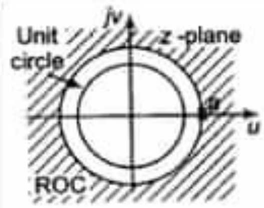
Property	Discrete Time Signal	z-Transform
Multiplication by n^m (or differentiation in z-domain)	$n^m x(n)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
Scaling in z-domain (or multiplication by a^n)	$a^n x(n)$	$X(a^{-1}z)$
Time reversal	x(-n)	$X(z^{-1})$
Conjugation	$x^*(n)$	$X^*(z^*)$
Convolution	$x_1(n)^* x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)x_2(n-m)$	$X_1(z)X_2(z)$

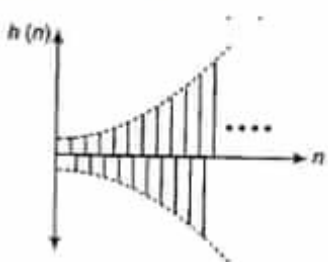
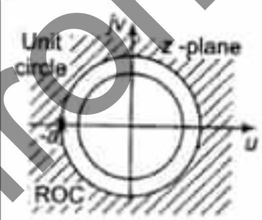
Property	Discrete Time Signal	z-Transform
Correlation	$r_{xy}(m)=\sum_{m=-\infty}^{+\infty} x(n)y(n-m)$	$X(z)Y(z^{-1})$
Initial value	$x(0)=\lim_{z\to\infty} X(z)$	
Final value	$x(\infty)=\lim_{z\to 1}(1-z^{-1})X(z) = \lim_{z\to 1}\frac{(z-1)}{z}X(z)$ If $X(z)$ is analytic for $\lvert z\rvert>1$	
Complex convolution theorem	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_c X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$
Parseval's relation	$\sum x_1(n)x_2^*(n)=\frac{1}{2\pi j}\int_c X_1(z)X_2^*\left(\frac{1}{z^*}\right)z^{-1}dz$	

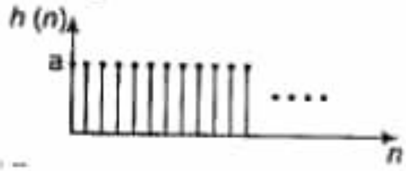
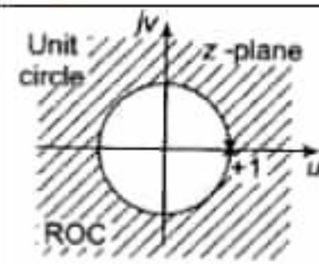
Impulse Response and Location of Poles

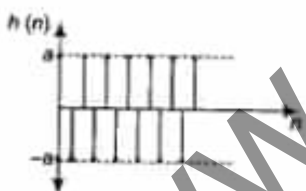
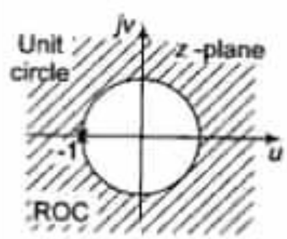
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = a^n u(n); 0 < a < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; stable system	$H(z) = \frac{z}{z-a}$ ROC is $\lvert z \rvert > a$ Pole at z = a	 Since 0 < a < 1, the pole z = a, lies inside the unit circle. The ROC contains the unit circle.

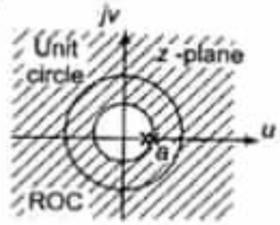
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = (-a)^n u(n); 0 < \lvert -a \rvert < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; stable system	$H(z) = \frac{z}{z+a}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ Pole at z = -a	 Since 0 < |-a| < 1, the pole z = -a, lies inside the unit circle. The ROC contains the unit circle.

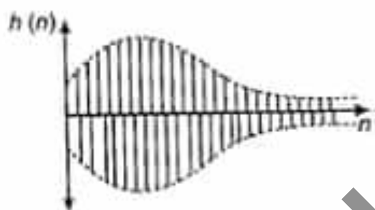
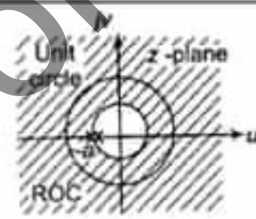
Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = a^n u(n); a > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{z}{z-a}$ ROC is $\lvert z \rvert > a$ Pole at $z = a$	 Since a > 1, the pole z = +a, lies outside the unit circle. The ROC does not contain the unit circle.

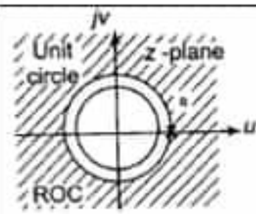
Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = -a^n u(n); \lvert -a \rvert > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{z}{z-a}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ Pole at $z = -a$	 Since 0 <|-a| > 1, the pole z = -a, lies outside the unit circle. The ROC does not contain the unit circle.

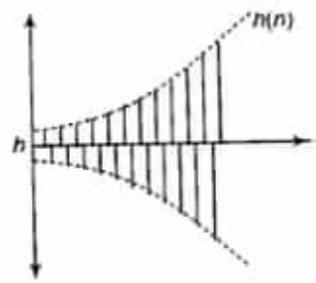
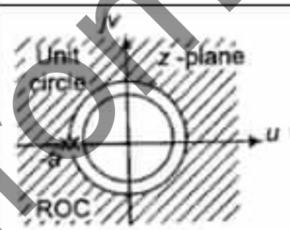
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = au(n); a > 0$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{az}{z-1}$ ROC is $\lvert z \rvert > 1$ Pole at $z = 1$	 The pole $z = 1$, lies on the unit circle. The ROC does not contain the unit circle.

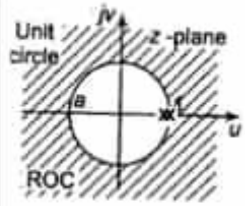
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = a(-1)^n u(n); a > 0$ (i.e., a is positive)  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{az}{z+1}$ ROC is $\lvert z \rvert > 1$ Pole at $z = -1$	 The pole $z = -1$, lies on the unit circle. The ROC does not contain the unit circle.

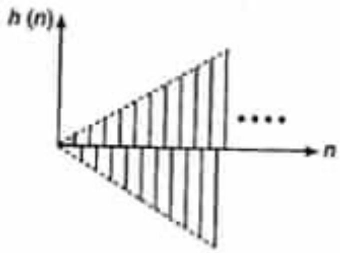
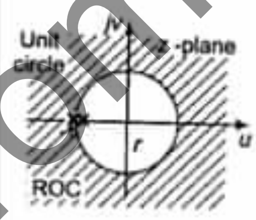
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = na^n u(n); 0 < a < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; stable system	$H(z) = \frac{az}{(z-a)^2}$ ROC is $\lvert z \rvert > 1$ Two Pole at z = a	 Since 0 <a < 1, the two poles at z = a, lie inside the unit circle. The ROC contains the unit circle.

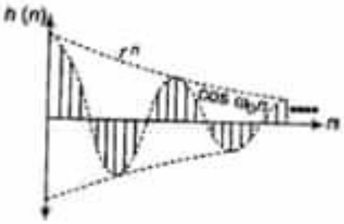
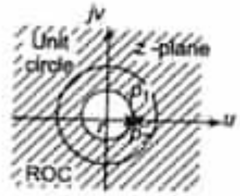
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = n(-a)^n u(n); 0 < \lvert -a \rvert < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; stable system	$H(z) = \frac{az}{(z+a)^2}$ ROC is $\lvert z \rvert > 1$ Two Pole at z = -a	 Since 0 <|-a| < 1, the two poles at z = -a, lie inside the unit circle. The ROC contains the unit circle.

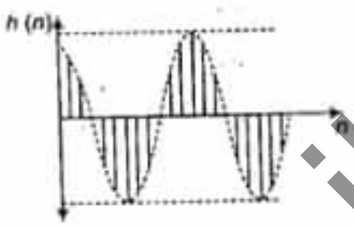
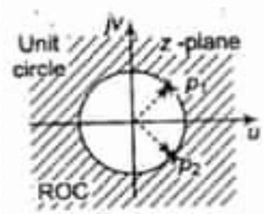
Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = na^n u(n); a > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty;$ unstable system	$H(z) = \frac{az}{(z-a)^2}$ ROC is $\lvert z \rvert > a$ Two Pole at z = a	 Since a > 1, the two poles at z = a, lie outside the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = n(-a)^n u(n); \lvert -a \rvert > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{az}{(z+a)^2}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ Two Pole at z = -a	 Since $\lvert -a \rvert > 1$, the two poles at z = -a, lie outside the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = nu(n)$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{z}{(z-1)^2}$ ROC is $\lvert z \rvert > 1$ Two Pole at $z = 1$	 The two pole $z = 1$, lie on the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	**Transfer Function**	**Location of Poles in z-plane and ROC**
$h(n) = n(-1)^n u(n); \lvert -a \rvert > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z) = \frac{z}{(z+1)^2}$ ROC is $\lvert z \rvert > 1$ Two Pole at $z = -1$	 The two pole $z = -1$, lie on the unit circle. The ROC does not contain the unit circle.

Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = r^n \cos\omega_0 n u(n); 0 < r < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; stable system	$H(z)$ $= \frac{z(z - r\cos\omega_0)}{(z - r\cos\omega_0)}$ $jr\sin\omega_0)$ $(z - r\cos\omega_0$ $+ jr\sin\omega_0)$ ROC is $\lvert z \rvert > r$ A pair of conjugate poles at $z = P_1 = r\cos\omega_0 + jr\sin\omega_0$ $z = P_2 = r\cos\omega_0 - jr\sin\omega_0$	 Since $0 < r < 1$, the conjugate pole pairs lie inside the unit circle. The ROC contains the unit circle.

Impulse Response h(n)	Transfer Function	Location of Poles in z-plane and ROC
$h(n) = \cos\omega_0 n u(n)$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; unstable system	$H(z)$ $= \frac{z(z - \cos\omega_0)}{(z - \cos\omega_0)}$ $j\sin\omega_0)$ $(z - \cos\omega_0$ $+ j\sin\omega_0)$ ROC is $\lvert z \rvert \geq r$ A pair of conjugate poles on unit circle at, $z = P_1 = \cos\omega_0 + j\sin\omega_0$ $z = P_2 = \cos\omega_0 - j\sin\omega_0$	 The conjugate pole pairs lie on the unit circle. The ROC does not contain the unit circle.