Fluid Mechanics

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Fluid Properties

Basic Concept

- A substance in the liquid / gas phase is referred to as 'fluid'.
- The distinction between a solid & fluid is made on the basis of the substance's ability to resist an applied shear (tangential) stress that tends to change its shape. A solid can resist an applied shear by deforming its shape whereas a fluid deforms continuously under the influence of shear stress, no matter how small is its shape. In solids, stress is proportional to strain, but in fluids, stress is proportional to 'strain rate.'

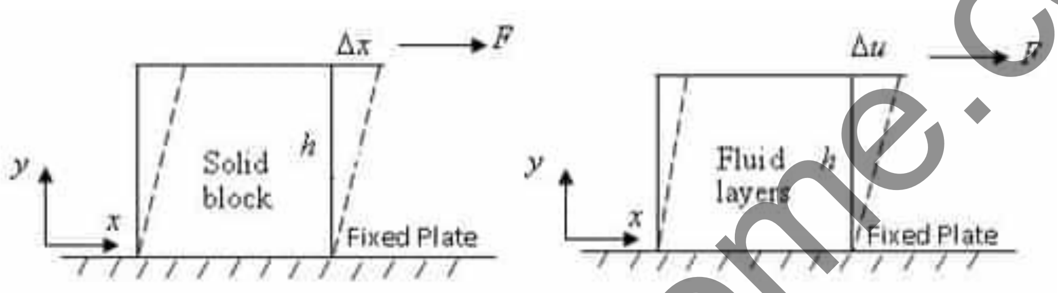


Illustration of solid and fluid deformation

Referring to Fig., the shear modulus of solid (S) and coefficient of viscosity (μ) for fluid can defined in the following manner;

$$S = \frac{\text{Shear stress}}{\text{Shear stain}} = \frac{F/A}{\Delta x/h} ; \mu = \frac{\text{Shear stress}}{\text{Shear strain rate}} = \frac{F/A}{\Delta u/h}$$

Here, the shear force (**F**) is acting on the certain cross-sectional area (**A**),

h is the height of the solid block / height between two adjacent layer of the fluid element,

Δx is the elongation of the solid block and Δu is the velocity gradient between two adjacent layers of the fluid.

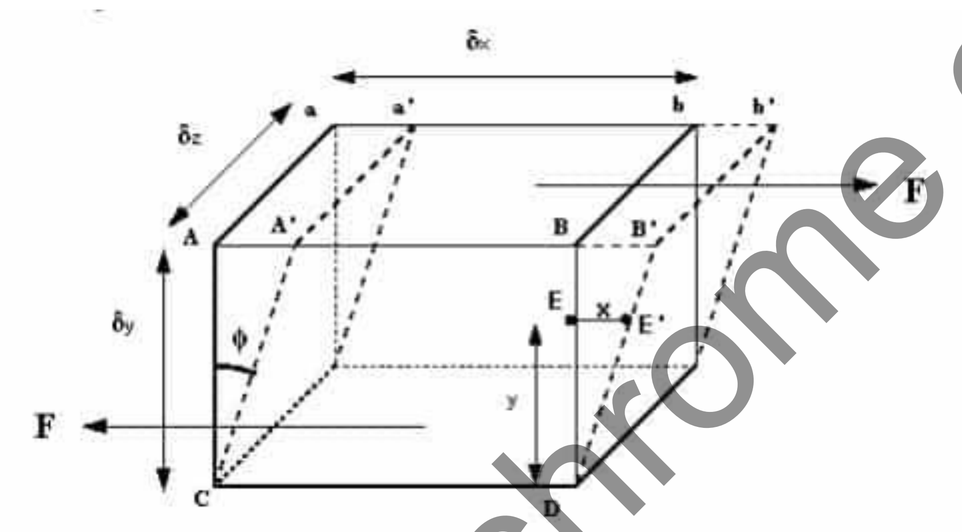
- So, a Fluid is a substance which deforms continuously, or flows, when subjected to shearing forces.
- If a fluid is at rest there are no shearing forces acting. All forces must be perpendicular to the planes which they are acting.

- Fluid can be treated as continuum and the properties at any point can be treated as bulk behavior of the fluids.

Newton's Law of Viscosity

- The shearing force F acts on the area on the top of the element. This area is given by A = δz × δx . We can thus calculate the shear stress which is equal to force per unit area i.e.

shear stress, τ = F/A



Fluid element under a shear force

- The deformation which this shear stress causes is measured by the size of the angle φ and is know as shear strain.
- **In a solid shear strain, φ, is constant for a fixed shear stress τ.**
- **In a fluid φ increases for as long as τ is applied - the fluid flows.**
- If the particle at point E (in the above figure) moves under the shear stress to point E' and it takes time t to get there, it has moved the distance x. For small deformations we can write

shear strain, φ = x/y

rate of shear strain =φ/t = x/ty = x/t.1/y = u/y

where x/t = u is the velocity of the particle at E

Using the experimental result that shear stress is proportional to rate of shear strain then

τ = Constant × u/y

The term u/y is the change in velocity with y, or the velocity gradient, and may be written in the differential form du/dy . The constant of proportionality is known as the dynamic viscosity, μ , of the fluid, giving

Fluids vs. Solids

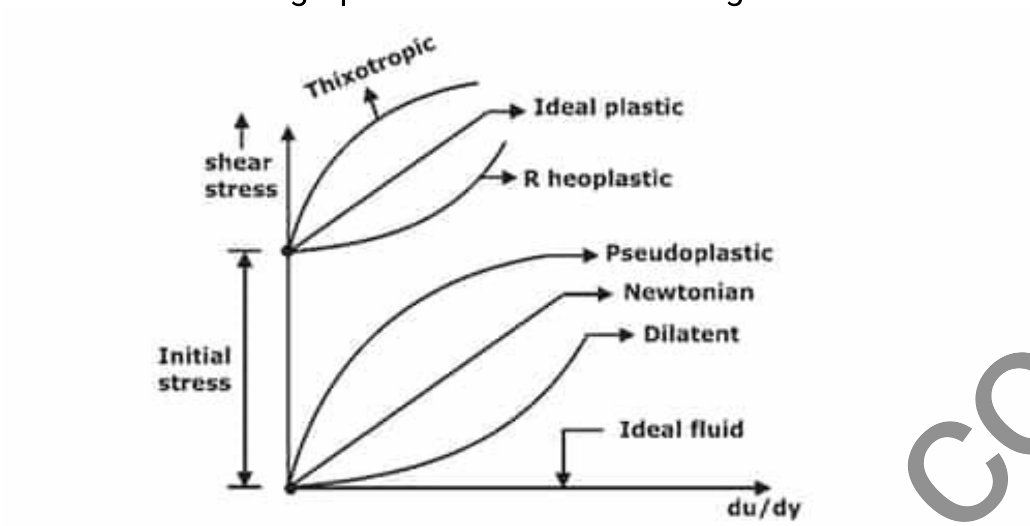
- For a solid the strain is a function of the applied stress (providing that the elastic limit has not been reached). For a fluid, the rate of strain is proportional to the applied stress.
- The strain in a solid is independent of the time over which the force is applied and (if the elastic limit is not reached) the deformation disappears when the force is removed. A fluid continues to flow for as long as the force is applied and will not recover its original form when the force is removed.

Newtonian / Non-Newtonian Fluids

- Fluids obeying Newton's law where the value of μ is **constant** are known as **Newtonian fluids**. If μ is constant the shear stress is linearly dependent on velocity gradient. This is true for most common fluids.
- Fluids in which the value of μ is **not constant** are known as **non-Newtonian fluids**.

Other types of Fluids

- There are several categories of these, and they are outlined briefly below. These categories are based on the relationship between shear stress and the velocity gradient (rate of shear strain) in the fluid. These relationships can be seen in the graph below for several categories.



Each of these lines can be represented by the equation

$$\tau = A + B\left(\frac{\delta u}{\delta y}\right)^n$$

where A, B and n are constants. For Newtonian fluids A = 0, B = μ and n = 1.

Below are brief description of the physical properties of the several categories

1. **Plastic**: Shear stress must reach a certain minimum before flow commences
2. **Bingham plastic**: As with the plastic above a minimum shear stress must be achieved. With this classification n = 1. An example is sewage sludge.
3. **Pseudo-plastic**: No minimum shear stress necessary and the viscosity decreases with rate of shear, e.g. colloidial substances like clay, milk, quicksand and cement.
4. **Dilatant substances**; Viscosity increases with rate of shear e.g. cornflour, printing inks and vinyl resin pastes.
5. **Thixotropic substances**: Viscosity decreases with length of time shear force is applied e.g. thixotropic jelly paints.
6. **Rheopectic substances**: Viscosity increases with length of time shear force is applied
7. **Viscoelastic materials**: Similar to Newtonian but if there is a sudden large change in shear they behave like plastic.
8. There is also one more - which is not real, it does **not exist** - known as the **ideal fluid**. This is a fluid which is assumed to have no viscosity.

Properties of Fluid

- Any characteristic of a system is called property. It may either be **intensive** (mass independent) or **extensive** (that depends on size of system). The state of a system is described by its properties. Most common properties of the fluid are:

1. **Pressure (p)**: It is the normal force exerted by a fluid per unit area. More details will be available in the subsequent section (Lecture 02). In SI system the unit and dimension of pressure can be written as, N/m^2 and $ML^{-1} T^{-2}$, respectively.
2. **Density**: The density of a substance is the quantity of matter contained in unit volume of the substance.

It is expressed in three different ways; **mass density** (**ρ = mass/volume**),

specific weight(ρg) and **relative density/specific gravity** water **SG = ρ/ρ_{water}**

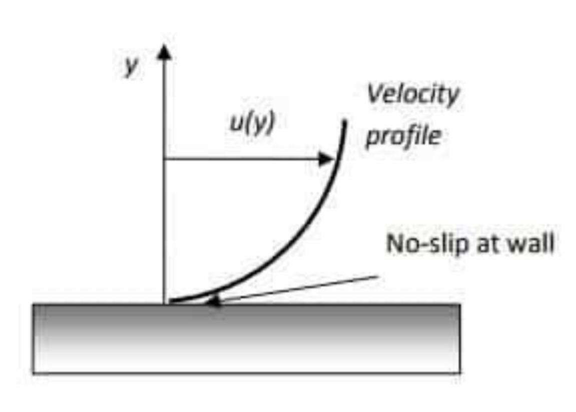
The units and dimensions are given as, For **mass density**; Dimension: $M L^{-3}$ Unit: **kg/m^3**

For **specific weight**; Dimension: $ML^{-2} T^{-2}$ Unit: **N/m^3**

The standard values for the **density of water** and **air** are given as **$1000kg/m^3$** and **$1.2\ kg/m^3$** , respectively. Many a times the reciprocal of mass density is called as specific volume (v).

3. **Temperature (T)**: It is the measure of hotness and coldness of a system. In thermodynamic sense, it is the measure of internal energy of a system. Many a times, the temperature is expressed in centigrade scale (°C) where the freezing and boiling point of water is taken as 0°C and 100°C, respectively. In SI system, the temperature is expressed in terms of absolute value in Kelvin scale (K = °C+ 273).

4. **Viscosity:** Viscosity is a measure of a fluid's resistance to flow. It determines the fluid strain rate that is generated by a given applied shear stress.



Velocity profile and shear stress

- A Newtonian fluid has a linear relationship between shear stress and velocity gradient:

$$\tau = \mu \frac{du}{dy}$$

This is known as **Newton's law of viscosity**.

- The shear stress is proportional to the slope of the velocity profile and is greatest at the wall.
- **The no-slip condition: at the wall velocity is zero relative to the wall. This is a characteristic of all viscous fluid.**
- The linearity coefficient in the equation is the **coefficient of viscosity**, **μ**(Ns/m^2), We can also use the **kinematic viscosity** v(m^2/s) =**μ/ρ**
- Temperature has a strong and pressure has a moderate effect on viscosity. The viscosity of **gases and most liquids increases** slowly with **pressure**.
- **Gas viscosity increases** with **temperature**. Two common approximations are the power law and the Sutherland law
- **Liquid viscosity decreases** with **temperature** and is roughly exponential.

5. **Thermal Conductivity(k)**: It relates the rate of heat flow per unit area (q)to the temperature gradient dT/dx and is governed by Fourier Law of heat conduction i.e.

q = -k.dT/dx

In SI system the unit and dimension of pressure can be written as, W/m.K and **MLT^{-3}** θ^{-1} , respectively

6. **Surface Tension:**

When a liquid and gas or two immiscible liquids are in contact, an unbalanced force is developed at the interface stretched over the entire fluid mass. The intensity of molecular attraction per unit length along any line in the surface is called as surface tension. For example, in a spherical liquid droplet of radius (r), the pressure difference (Δp) between the inside and outside surface of the droplet is given by,

$\Delta p = 2\sigma / r$

Reason: - Cohesive force b/w molecules.

Definition: - Force required to maintain unit length of the film in equilibrium, means force per unit length

Unit:- (N/m)

→ Due to surface tension

Increasing internal pressure of droplet.

The tendency of liquid droplet to attain minimum surface area at a given volume, only for this reason, shape of droplet is "Sphere".

NOTE:-

Minimum surface area at a given volume = surface area of sphere.

Dependency of surface tension:-

Temperature:-

If temperature increases, cohesive force decreases and this will results in decrease in surface tension

If continuous decreasing in temperature takes place than surface tension becomes zero at "critical point of temperature".

Additives or {impurities}

Surfactants:-

→ Reduce the surface tension

Ex. Organic solute

Some salt [NaCl] increase the surface tension

Curved surface indicate pressure difference (mean pressure jump)
Pressure higher on concave side (in given figure)

Pressure difference between p_i and p_0 for

Soap bubble:

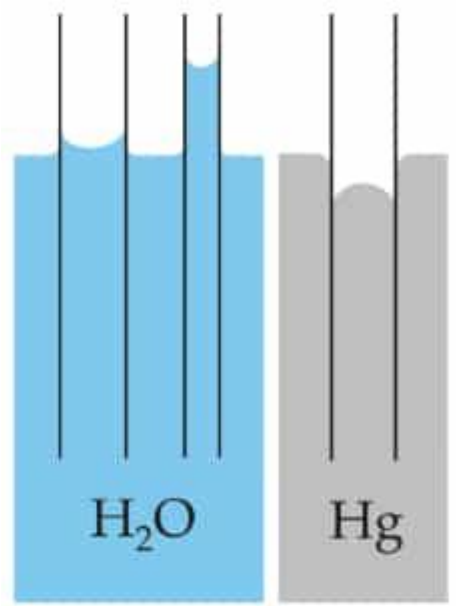
$$P_i - P_o = \left(\frac{4\sigma_s}{R}\right)$$

Water Droplet:

$$p_i - p_o = \left(\frac{2\sigma_s}{R}\right)$$

Where σ is surface tension and R is the radius of curvature for bubble or droplets.

7. **Capillary action:** Capillary action (sometimes capillarity, capillary motion, or wicking) is the ability of a liquid to flow in narrow spaces without the assistance of, or even in opposition to, external forces like gravity. It occurs because of intermolecular forces between the liquid and surrounding solid surfaces. If the diameter of the tube is sufficiently small, then the combination of surface tension (which is caused by cohesion within the liquid) and adhesive forces between the liquid and container wall act to propel the liquid.

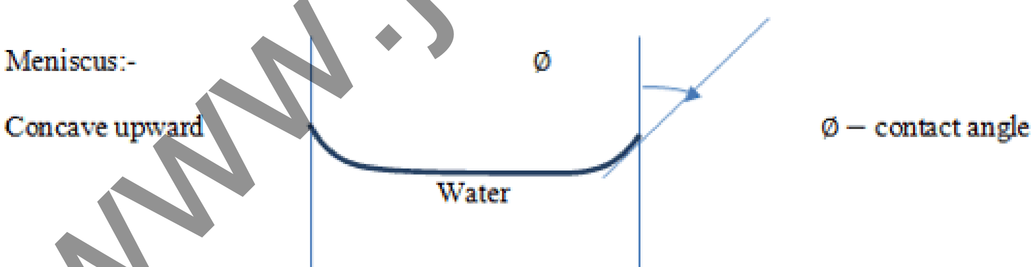


Capillary action of water compared to mercury, in each case with respect to a polar surface such as glass

Capillary Effect:

Reason:- Cohesive force or surface tension and Adhesive forces. (Both force responsible for Capillary effect)

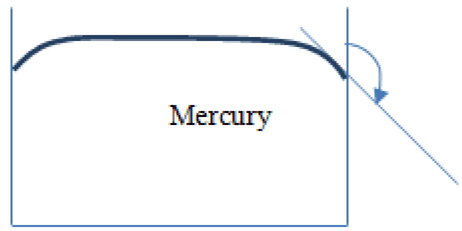
- Curved free surface inside the capillaries is called meniscus.
- Rise or fall of liquid inside the tube is due to contact angle b/w liquid surface and capillary tube.



NOTE: if $\varnothing < 90^\circ$ then

- Level of liquid inside the tube is rise
- Liquid is known as Wetting liquid
- In this case: cohesive force >adhesive force

Ø



→Meniscus: Convex Upward

If $\varnothing > 90^\circ$ then

- Level of liquid fall inside the tube
- liquid is known as Non-wetting liquid
- In this case:-

$$\begin{bmatrix} adhesive > cohesive \\ force \quad\quad force \end{bmatrix}$$

$\varnothing =$ Angle b/w tangent to the liquid surface and solid surface at the contact point.

Height of a meniscus

The height *h* of a liquid column is given by

$$h = \frac{2\gamma \cos\theta}{\rho g r},$$

where γ is the liquid-air surface tension (force/unit length), θ is the contact angle, **ρ is the density of liquid** (mass/volume), g is the local acceleration due to gravity (length/square of time), and **r is the radius of tube**.

Thus the thinner the space in which the water can travel, the further up it goes.

Observations

- For water –glass interface
  $\varnothing = 0$ So $\cos\varnothing = 1$ this results in

$$h = \frac{2\sigma_s}{\rho g R}$$

- Height of capillary rise is a function of

$$\left(h \propto \frac{1}{R}\right), \left(h \propto \frac{1}{\rho}\right)$$

- If diameter of tube > 1 cm than Capillary effect negligible

8. **Bulk Modulus of Elasticity:**

- Compressibility of liquid is measured by bulk modulus of elasticity.
- Bulk modulus is represented the compressive stress per unit volumetric strain.
- Bulk modules k

$$K = \frac{\Delta p}{(\Delta v / v)} \begin{cases} \Delta p \rightarrow Compressive\ stress \\ \frac{\Delta v}{v} \rightarrow Volumetric\ strain always -ive \end{cases}$$

- K → always positive or is a positive quantity having unit of pressure

$$\left(\frac{N}{m^2}\right).$$

- Truly incompressible substance: means
- $\frac{\Delta v}{v} = 0$

  So, K (bulk modulus) = ∞

Note:

K increase means Resistance to further compression increases.

- For liquid K **increases** with **decreases** in temperature: with decrease in temperature cohesive force between molecules increases, which results in higher resistance to further compression.
- For gases K **increases** with **increases** in temperature: With increase in temperature, collision between gas particle increases and results in higher internal pressure so the resistance to further compression increases.

9. **Vapour Pressure and cavitation:**

- Saturation Temperature:
  For a given pressure, the temperature at which a pure substance changes phase is known as saturation temperature
- Saturation Pressure:
  At a given temperature, the pressure at which a pure substance changes phase.

Example: at 1 atm pressure (const. pressure) saturation temperature is 100 c and at constant temp. 100 c saturation pressure for water is 1 atm.

Vapour Pressure:

- For liquid, pressure exerted by its vapour, in phase equilibrium with its liquid at a given temperature
- Vapour pressure increases [with temperature with increases and rate molecules escaping liquid surface increasing
- When vapour pressure equal to pressure on the liquid – boiling occur.

Cavitation:

- Cavitation is a phenomenon which occurs in a liquid flow system.
- If liquid undergo pressure below vapour pressure during flow, than sudden vaporization takes place
- Vapour bubbles collapse as they are swept from the low pressure region, generating highly destructive pressure waves.
- Cavitation can also occur if a liquid contains dissolved air or other gases, {Reason-**Solubilities** Decrease with decreasing pressure}
- Risk of cavitation is greater at higher temperatures.

Example: Given a flow system (water) and Temperature is 36 c. Find the minimum pressure to avoid cavitation?

Solution: Minimum pressure to avoid cavitation is equal to vapour pressure of that liquid at given temperature for water

$$P_{min} = P_v = 4.25 \text{ kPa} \qquad (T = 30°c)$$

Note:

1. Partial pressure is the pressure exerted by a component in a mixture of gases.

2. for pure substance vapour pressure and saturation pressure, both are equal.
3. If external pressure is equal to or less than the vapour pressure, boiling of liquid will start no matter how much temperature.

Manometry and Buoyancy & Hydrostatic Forces on Surface

Manometry and Buoyancy

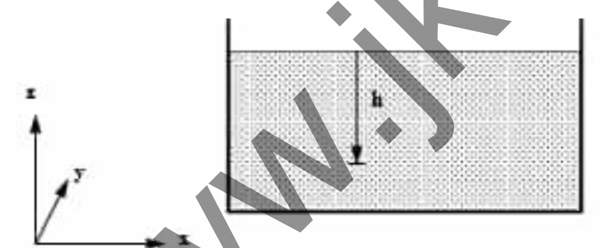
Manometry

- The pressure is proportional to the height of a column of fluid.
- Manometry is the field of science which deals with the evaluation of the pressure of the fluid.
- The instrument used to carry out the complete process is termed a Manometer.
- Types of Manometers: **Barometer, Piezometer and U-tube Manometer**.

Manometers use the relationship between pressure and head to measure pressure.

Relation between Hydrostatic pressure & Head

We have the vertical pressure relationship **p = ρgz + constant** measuring z from the free surface so that z = -h and surface pressure is atmospheric,p_{atm}



$$p = \rho g h + \text{constant}$$

$$p_{\text{atmospheric}} = \text{constant}$$

so

$$p = \rho g h + p_{\text{atmospheric}}$$

We generally assume atmospheric pressure as the datum,

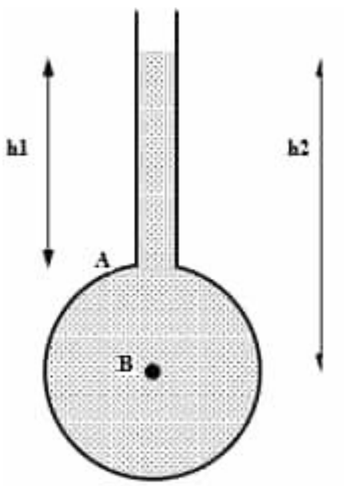
Gauge pressure, $p_g = \rho gh$

The lower limit of any pressure is the pressure in a perfect vacuum. Pressure measured above a perfect vacuum (zero) is known as absolute pressure.

Absolute pressure, $p_a = \rho gh + p_{atmospheric}$

Absolute pressure = Gauge pressure + Atmospheric

Piezometer Tube Manometer



- The simplest manometer is an open tube. This is attached to the top of a container with liquid at pressure. containing liquid at a pressure.
- The tube is open to the atmosphere, The pressure measured is relative to atmospheric so it measures gauge pressure.
- Pressure at A = pressure due to column of liquid h_1

$p_a = \rho gh_1$

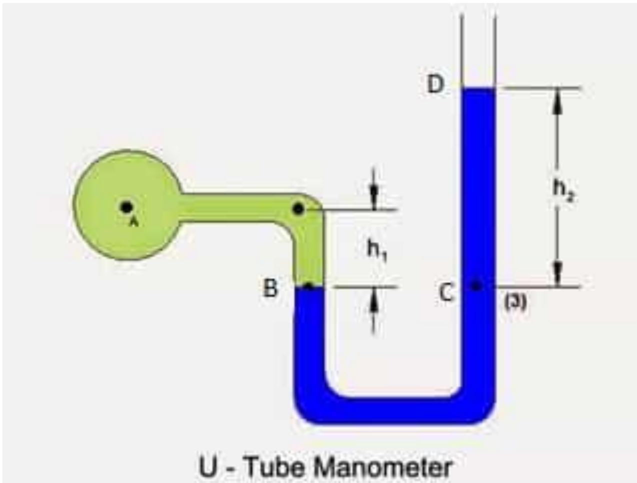
- Pressure at B = pressure due to column of liquid h_2

$P_b = \rho gh_2$

Limitations of Piezometer:

- Can only be used for liquids
- Pressure must above atmospheric
- Liquid height must be convenient i.e. not be too small or too large

U-tube Manometer



U - Tube Manometer

- It consists of a U shaped bend whose one end is attached to the gauge point 'A' and the other end is open to the atmosphere.
- It can measure both positive and negative (suction) pressures.
- "U"-Tube enables the pressure of both liquids and gases to be measured "U" is connected as shown and filled with manometric fluid.

Note:

- The manometric fluid density should be greater than of the fluid measured, $\rho_{man} > \rho$
- The two fluids should not be able to mix they must be immiscible.
- Pressure in a continuous static fluid is the same at any horizontal level, pressure at B = pressure at C

$$P_B = P_C$$

- For the left-hand arm pressure at B
    - pressure at A + pressure of height of liquid being measured

$$P_B = P_A + \rho g h_1$$

- For the right-hand arm pressure at C =
    - pressure at D + pressure of height of manometric liquid

$$P_C = P + \rho_{mano} g h_2$$

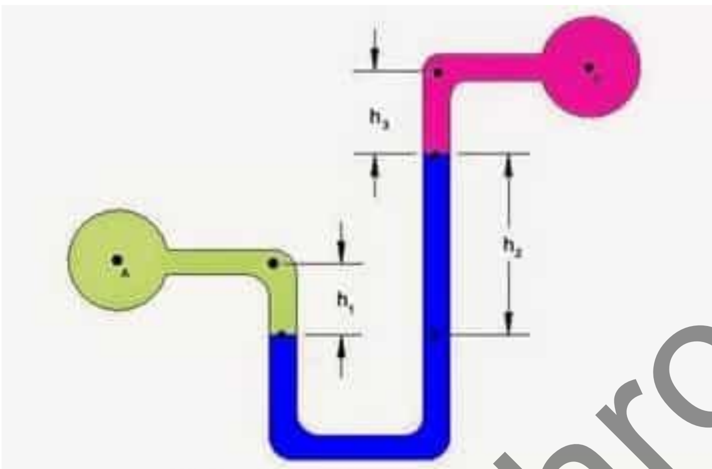
We are measuring gauge pressure we can subtract $p_{atmospheric}$ giving

$P_B = P_C$

$P_A = P + \rho_{mano} g h_2 - \rho g h_1$

Differential U-Tube Manometer

- A U-Tube manometric liquid is heavier than the liquid for which the pressure difference is to be measured and is not immiscible with it.



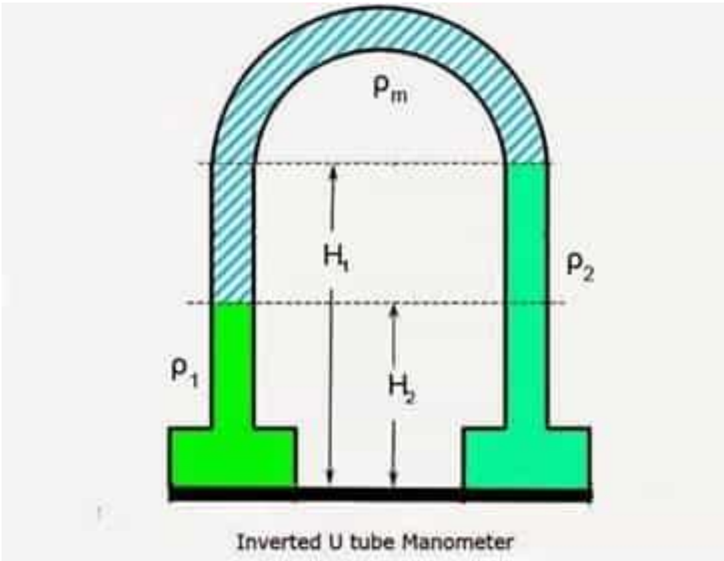
Differential U - Tube Manometer

- The pressure difference between A and B is given by equation

$P_A - P_B = \rho_2 h_2 + \rho_3 h_3 - \rho_1 h_1$

Inverted U-Tube Manometer

- Inverted U-Tube manometer consists of an inverted U Tube containing a light liquid.
- This is used to measure the differences of low pressures between two points where better accuracy is required.
- It generally consists of an air cock at top of the manometric fluid type.



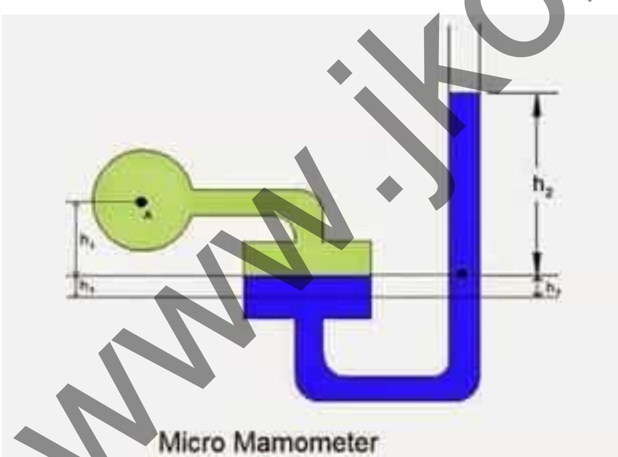
Inverted U tube Manometer

Pressure difference can be calculated from equation:

$$P_1 - \rho_1 g H_2 - \rho_m g(H_1 - H_2) = P_2 - \rho_2 g H1$$

Micro Manometer

- Micro Manometer is the modified form of a simple manometer whose one limb is made of larger cross sectional area.
- It measures very small pressure differences with high precision.



Micro Mamometer

Let 'a' = area of the tube, A = area of the reservoir, h_3 = Falling liquid level reservoir,

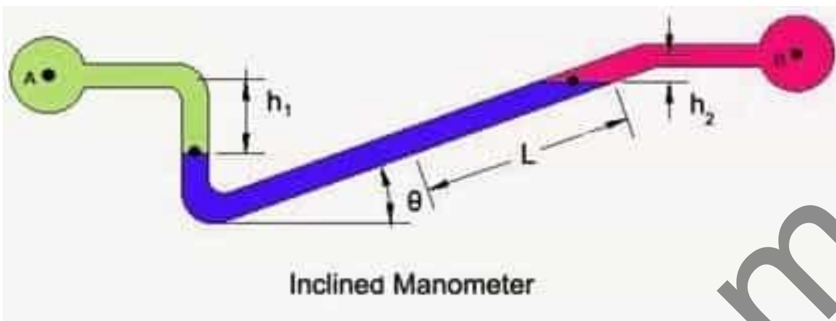
h_2 = Rise of the liquid in the tube,

- By Volume Equality, $\mathbf{Ah_3 = ah_2}$
- Equating pressure heads at datum,

$$P_1 = (\rho_m - \rho_1)gh_3 + \rho_m gh_2 - \rho_1 gh_1$$

Inclined Manometer

- An inclined manometer is used for the measurement of small pressures and is to measure more accurately than the vertical tube type manometer.
- Due to inclination, the distance moved by the fluid in manometer is more.

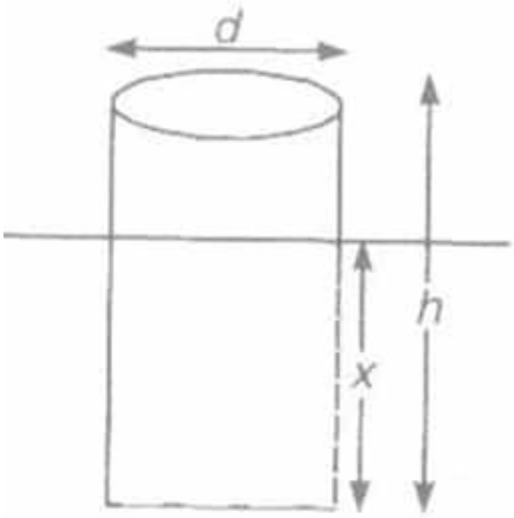


- Pressure difference between A and B is given by equation

$$P_A - P_B = \rho_{2L\sin\Theta} + \rho_3 h_2 - \rho_1 h_1$$

Buoyancy

Buoyancy is also known as buoyant force. It is the force exerted on an object that is wholly or partly immersed in a fluid.



Cylinder-fluid system

Concept of Buoyancy: When a body is immersed in a fluid, an upward force is exerted by fluid on the body which is equal to weight of fluid displaced by body. This acts as upward.

Archimedes' Principle: It states, when a body is immersed completely or partially in a fluid, it is lifted up by a force equal to weight of fluid displaced by the body.

Buoyant force = Weight of fluid displaced by body

Buoyant force on cylinder =Weight of fluid displaced by cylinder

V_{Sm} = Value of immersed part of solid or Volume of fluid displaced

F_B = P_{water} X g X Volume of cylinder immersed inside the water

= $p_w g \frac{\pi}{4} d^2$ (w = mg = pVg)

Principle of Flotation: According to this principle, if weight of body is equal to buoyant force then, body will float.

F_B = mg

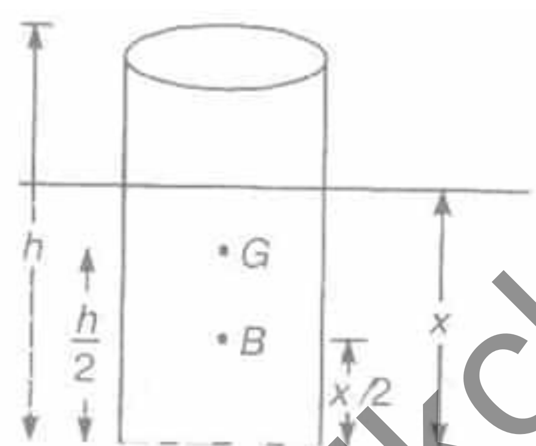
$V_{s_m} plg = V_s p_s g$

$$p_w g \frac{\pi}{4} d^2 x = p_{cylinder} g \frac{\pi}{4} d^2 h$$

$$p_w x = p_{cylinder} h$$

- The factors that affect buoyancy are: the density of the fluid, the volume of the fluid displaced, and the local acceleration due to gravity.
- The buoyant force is not affected by the mass of the immersed object or the density of the immersed object.

Center of Buoyancy: The point at which force of buoyancy acts is called **center of buoyancy.** It lies on center of gravity of volume of fluid displaced or center of gravity of the part of the body which is inside the water. Point B is the center of buoyancy.



Centre of buoyancy diagram

Buoyancy on a submerged body:

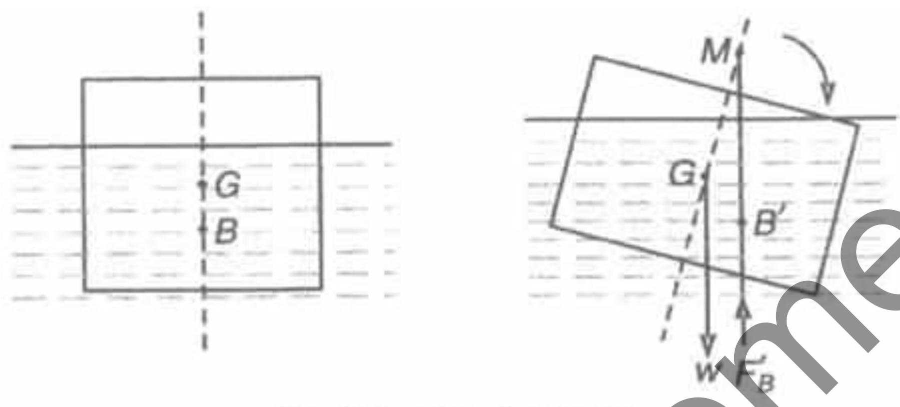
- The Archimedes principle states that the buoyant force on a submerged body is equal to the weight of liquid displaced by the body, and acts vertically upward through the centroid of the displaced volume.
- Thus the net weight of the submerged body, (the net vertical downward force experienced by it) is reduced from its actual weight by an amount that equals the buoyant force.

Buoyancy on a partially immersed body:

- According to Archimedes principle, the buoyant force of a partially immersed body is equal to the weight of the displaced liquid.

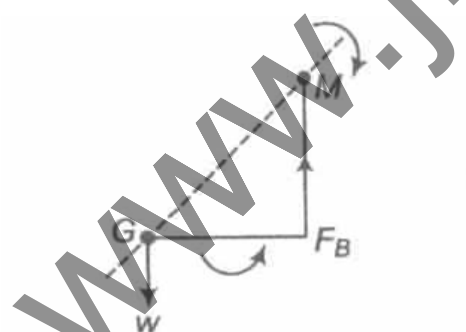
- Therefore the buoyant force depends upon the density of the fluid and the submerged volume of the body.
- For a floating body in static equilibrium and in the absence of any other external force, the buoyant force must balance the weight of the body.

Metacentre of a Floating Body: If a body that is floating in liquid is given a small angular displacement, it starts oscillating about some point M. This point is called the **metacentre.**



Body floating in liquid

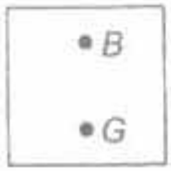
The equilibrium of a submerged body in a liquid requires that the weight of the body acting through its centre of gravity should be colinear with equal hydrostatic lift acting through the centre of buoyancy. Let us suppose that a body is given a small angular displacement and then released. Then it will be said to be in distance MG is called **metacentric height** (it is the distance between the gravity centre and metacentre)



Gravity centre and metacentre

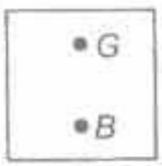
Stability of Submerged Body: *It is classified into three groups.*

- **Stable Equilibrium:** When the centre of buoyancy lies above the centre of gravity, the submerged body is stable.



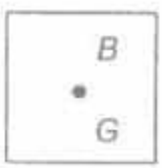
Stable equilibrium

- **Unstable Equilibrium:** When B lies below G, then body is in unstable equilibrium.



Unstable equilibrium

- **Neutral Equilibrium:** When B and G coincide then, body is in neutral equilibrium.



Neutral equilibrium

Stability of Floating Bodies: When the body undergoes an angular displacement about a horizontal axis, the shape of the immersed volume changes and so the centre of buoyancy moves relative to the body.

- **Stale Equilibrium:** When a body is given a small angular displacement by external means and if body comes to its original position due to internal forces then, it is called stable equilibrium.

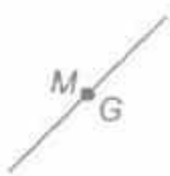
Stable position

It occurs, when metacentre lies above centre of gravity.

- **Unstable Equilibrium:** In the above case, if body does not come in its original position and moves further away then, it is known as unstable equilibrium. M lies below centre of gravity.

Unstable position

- **Neutral equilibrium:** When a body is given a small angular displacement and it sets on new position then, body is called in neutral equilibrium. In this, M and G coincide.



Neutral position

- Relation between B,G and M is

GM = I/V - BG

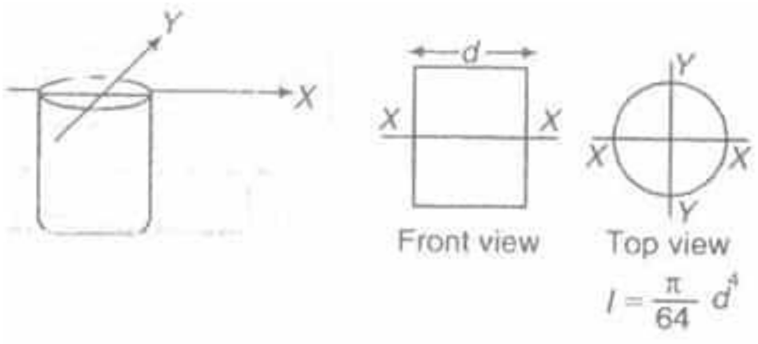
Here, **I** = Least moment of inertia of plane of body at water surface

$$I = \min(I_{xx}, I_{yy}), I_{xx} = \frac{bd^3}{12}, I_{yy} = \frac{bd^3}{12},$$

G = Centre of gravity

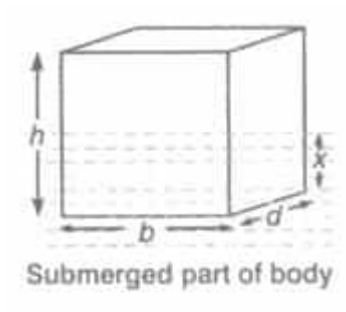
B = Centre of buoyancy

M = Metacentre



V is volume submerged inside the water can be given as $V = bdx$

Where b,d and x are the length, width and depth of the section or body.



- BG is the distance between centre of gravity and centre of buoyancy. (In other words, BG=distance between centre of gravity of whole body and centre of gravity of submerged part of body)
- When we find out GM then, we can determine the status of body as
    - GM > 0 (stable equilibrium),
    - GM < 0 (unstable equilibrium),
    - GM = 0 (neutral equilibrium)

Hydrostatic Force on Surfaces

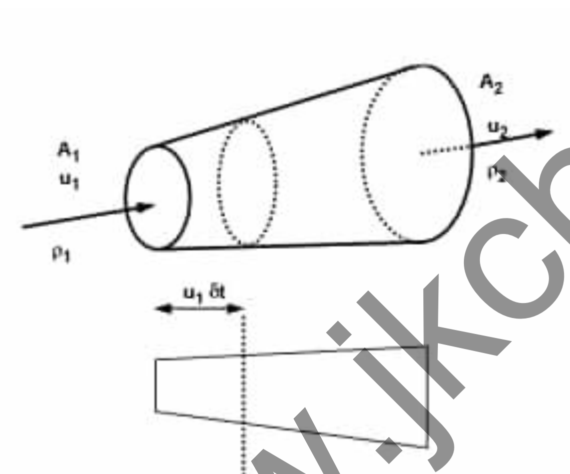
Fluid Statics

- Fluid Statics deals with fluids at rest while Fluid Dynamics studies fluids in motion.
- Any force developed is **only due to normal stresses** i.e, pressure. Such a condition is termed the hydrostatic condition.
- Fluid Statics is also known as Hydrostatics.

- A static fluid can have **no shearing force** acting on it, and that any force between the fluid and the boundary must be acting at **right angles** to the boundary.
- For an element of fluid at rest, the element will be in equilibrium. The sum of the components of forces in any direction will be zero. The sum of the moments of forces on the element about any point must also be zero.
- Within a fluid, the pressure is same at all the points in all the directions.
- Pressure at the wall of any vessel is perpendicular to the wall
- Pressure due to depth is $P = \rho gh$, and is the same at any horizontal level of connected fluid.

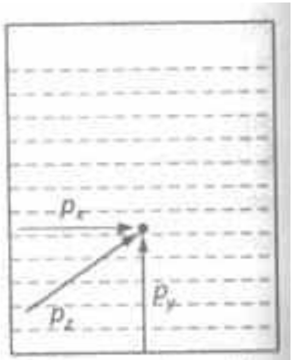
Fluid Pressure at a Point

- If a fluid is **Stationary**, then force acting on any surface or area is perpendicular to that surface.
- If the force exerted on each unit area of a boundary is the same, the pressure is said to be uniform



Pascal's Law for Pressure At A Point

- It states that pressure or intensity of pressure at a point in a static fluid (fluid is in rest) is equal in all directions. If fluid is not in motion then according to Pascal's law,

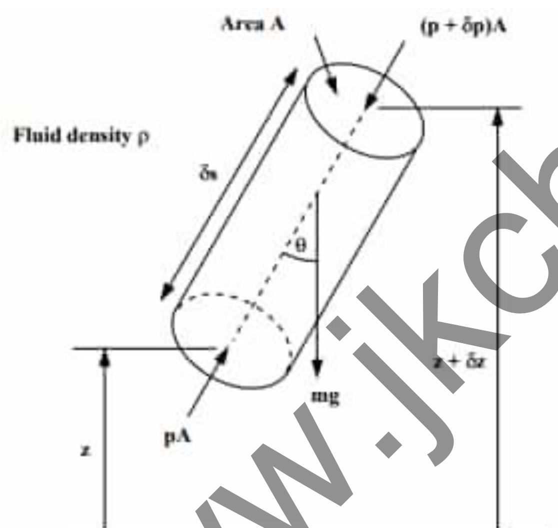


Pressure at a point

$p_x = p_y = p_z$

where, p_x, p_y and p_z are the pressure at point x,y,z respectively.

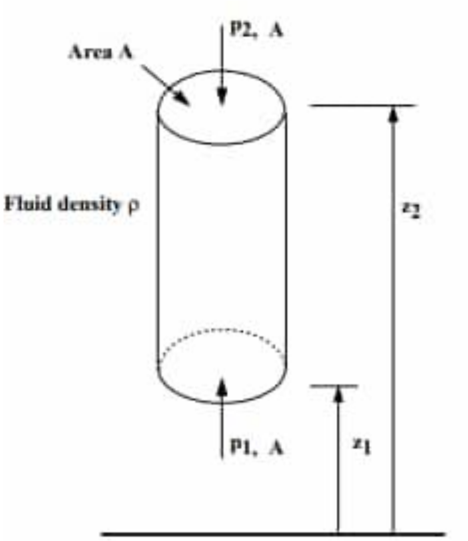
General Equation For Variation Of Pressure in a Static Fluid



(A cylindrical element of fluid at an arbitrary orientation)

$$\frac{dp}{ds} = -\rho g \cos\theta$$

Vertical Variation Of Pressure in a Fluid Under Gravity



Taking upward as positive, we have

Vertical cylindrical element of fluid cross sectional area = A

mass density = ρ

The forces involved are:

- Force due to p_1 on A (upward) = p_1. A
- Force due to p_2 on A (downward) = p_2. A
- Force due to weight of element (downward) = mg

= mass density x volume x g

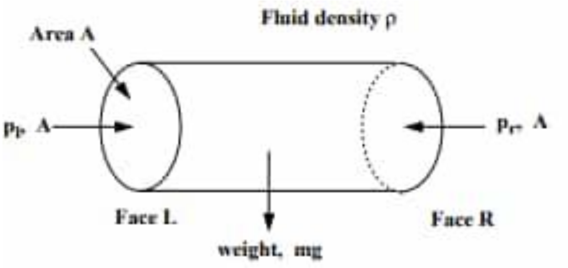
= $\rho.g.A.(z_2 - z_1)$

Thus in a fluid under gravity, **pressure decreases linearly with increase in height**

$$p_2 - p_1 = \rho g A(z_2 - z_1)$$

This is the hydrostatic pressure change.

Equality Of Pressure At The Same Level In A Static Fluid



Horizontal cylindrical element cross sectional area = A

mass density =ρ

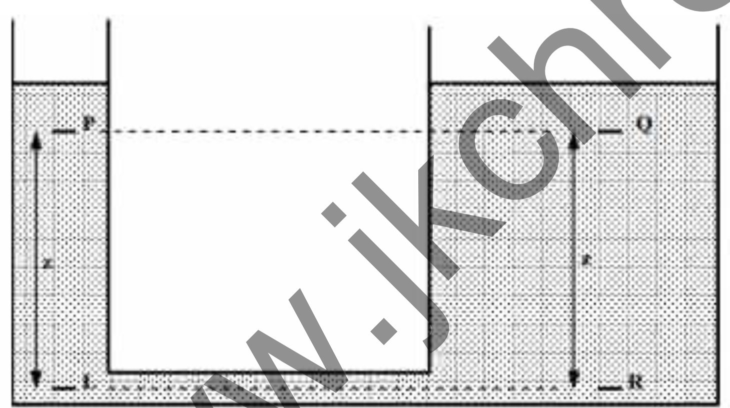
left end pressure = p_l

right end pressure = p_r

For equilibrium, the sum of the forces in the x direction is zero= $\mathbf{p_l . A = p_r . A}$

$\mathbf{p_l = p_r}$

So, **Pressure in the horizontal direction is constant.**



As we know, $p_l = p_r$

For a vertical pressure change we have

$$p_l = p_p + \rho g z$$

and

$$p_r = p_q + \rho g z$$

so

$$p_p + \rho g z = p_q + \rho g z$$

$$p_p = p_q$$

Thus, pressure at the two equal levels is the same.

Total Hydrostatic Force on Plane Surfaces

- For horizontal plane surface submerged in liquid, or plane surface inside a gas chamber, or any plane surface under the action of uniform hydrostatic pressure, the total hydrostatic force is given by

F = p. A

where *p* is the uniform pressure and *A* is the area.

- In general, the total hydrostatic pressure on any plane surface is equal to the product of the area of the surface and the unit pressure at its center of gravity.

F = p_{cg}**. A**

where p_{cg} is the pressure at the center of gravity.

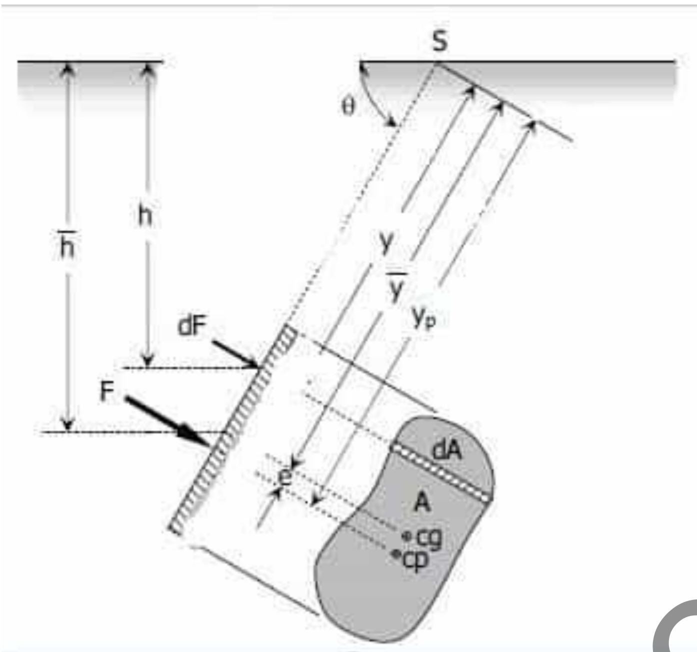
- For homogeneous free liquid at rest, the equation can be expressed in terms of unit weight γ of the liquid.

F=γh'A

where **h'** is the depth of liquid above the centroid of the submerged area.

Derivation of Formulas(Not required for exam)

The figure shown below is an inclined plane surface submerged in a liquid. The total area of the plane surface is given by *A*, *cg* is the center of gravity, and *cp* is the center of pressure.



(Forces on a inclined plane surface)

The differential force *dF* acting on the element *dA* is

$dF=p.\ dA$

$dF=\gamma.\ h.\ dA$

From the figure

$h=y\sin\theta,$
$dF=\gamma.(y\sin\theta).\ dA$

Integrate both sides and note that *γ* and *θ* are constants,
$F=\gamma.\ \sin\theta.\ \int y.dA$

So, $F=\gamma.\ \sin\theta.\ \int y.dA$

Recall from Calculus that

$\int y.dA = A.\bar{y}$

Hence, $F = (\gamma.\sin\theta)A.\bar{y}$

$F = \gamma.(\bar{y}\sin\theta).A$

From the figure, $\bar{y}\sin\theta = \bar{h}$, thus,

$F = \gamma\bar{h}A$

The product $\gamma\bar{\bar{h}}$ is a unit pressure at the centroid at the plane area, thus, the formula can be expressed in a more general term below:

$F = p_{cg.}A$

Location of Total Hydrostatic Force (Eccentricity)

From the figure above, *S* is the intersection of the prolongation of the submerged area to the free liquid surface. Taking moment about point *S*.

$Fy_p = \int y.dF$

Where
$dF = \gamma(y\sin\theta)dA$
$F = \gamma(\bar{y}\sin\theta)A$

$[\gamma(\bar{y}\sin\theta)A]y_p = \int y[\gamma(y\sin\theta)dA][\gamma(\bar{y}\sin\theta)A]y_p$

$= \int y[\gamma(y\sin\theta)dA]$

$(\gamma\sin\theta)A\bar{y}y_p = (\gamma\sin\theta)\int y^2dA(\gamma\sin\theta)A\bar{y}y_p$

$= (\gamma\sin\theta)\int y^2dA$

$A\bar{y}y_p = \int y^2dA$

Again from Calculus, $\int y^2dA$ is called moment of inertia denoted by *I* Since our reference point is *S*,

$A\bar{y}y_p = I_S$

Thus,

$y_p = I_S/A\bar{y}$

By transfer formula for moment of inertia $I_S = I_g + A\bar{y}^2$, the formula for y_p will become

$y_p = (I_g + A\bar{y}^2)/A\bar{y}$ or

$y_p = \bar{y} + I_g/A\bar{y}$

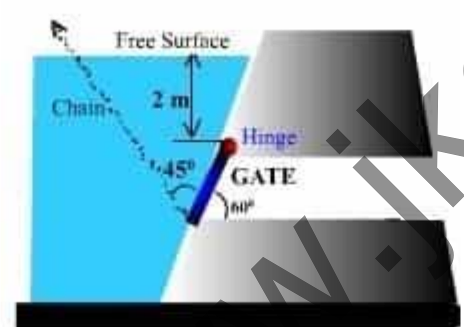
From the figure above, $y_p = \bar{y} + e$, thus, the distance between cg and cp is

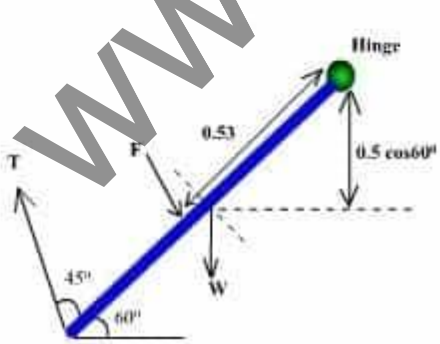
Eccentricity, $e = I_g/A\bar{y}$

Example 1

An opening in a dam is covered with a plate of 1 m square and is hinged on the top and inclined at 60 ^{0} to the horizontal. If the top edge of the gate is 2 m below the water level what is the force required to open the gate by pulling a chain set at 45 ^{0} angle with the plate and set to the lower end of the plate. The plate weighs 2200 N.

Solution:





Free body diagram

Given data: Area of the gate= 1 m^2

Total force on the gate= $F = wA\bar{x}$ where $\bar{x} = 2 + \frac{1}{2}\sin 60^0$

Depth of the center of pressure $\bar{h} = \bar{x} + \frac{I_G \sin^2 60^0}{A\bar{x}}$

Distance of the application point of the force from the hinge $= (\bar{h} - 2).\frac{1}{\sin 60^0} = 0.53m$

Taking moments about the hinge,

$T.1\sin 45^0 = F \times 0.53 + 2200 \times \frac{1}{2}\cos 60^0$

Then, T = 18.66

Answer: 18.66 kN

Total Hydrostatic Force on Curved Surfaces:

- In the case of curved surface submerged in liquid at rest, it is more convenient to deal with the horizontal and vertical components of the total force acting on the surface. Note: the discussion here is also applicable to plane surfaces.
- **Horizontal Component:** The horizontal component of the total hydrostatic force on any surface is equal to the pressure on the vertical projection of that surface.

$F_H = p_{cg}. A$

- **Vertical Component:** The vertical component of the total hydrostatic force on any surface is equal to the weight of either real or imaginary liquid above it.

$F_V = \gamma . V$

- **Total Hydrostatic Force:**

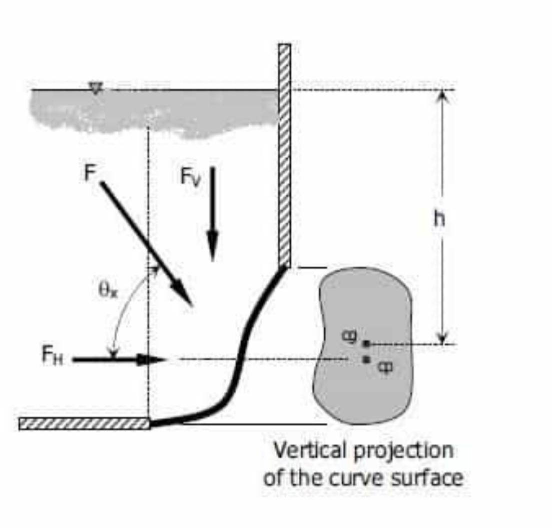
$F = \sqrt{(F_H^2 + F_V^2)}$

- **Direction of F:**

$\tan\theta_x = F_V / F_H$

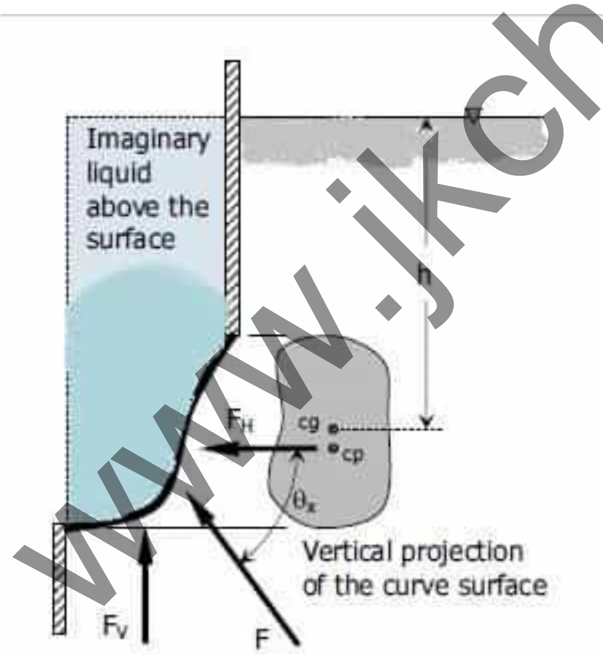
Case 1: Liquid is above the curve surface

The vertical component of the hydrostatic force is downward and equal to the volume of the real liquid above the submerged surface.



Case 2: Liquid is below the curve surface

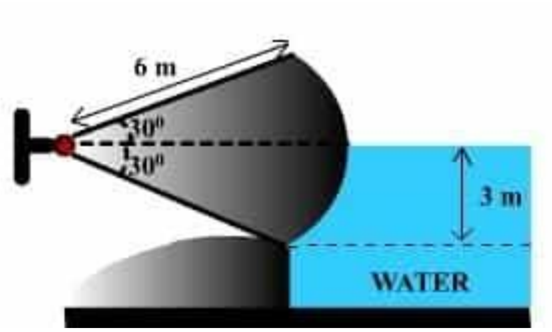
The vertical component of the hydrostatic force is going upward and equal to the volume of the imaginary liquid above the surface.



Example 2

The length of a tainter gate is 1 m perpendicular to the plane of the paper. Find out the total horizontal force on the gate and the total hydrostatic force on the gate.

Solution:



Horizontal hydrostatic force on the tainter gate

$$F_h = wA\bar{x}$$
$$= 44.145 \text{ KN.}$$

where projected area A = 3×1 = 3 m^2.

The vertical force is equal to the weight of water displaced by the shaded area.

The area of the shaded portion $= \frac{\pi}{12}\times 6^2 - \frac{1}{2}\times 3\sqrt{3}\times 3 = 1.63 m^2$.

The vertical force, $F_v = 9810\times 1.63\times 1 = 15.99$ KN.

The resultant force, $F = \sqrt{F_v^2 + F_h^2}$ and $\tan\theta = F_v / F_h$.

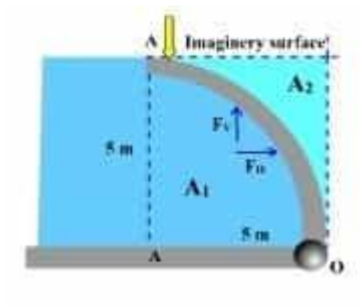
Answer: 46.95 KN and

q = 19.9^0 With horizontal

Example 3:

A quarter circle (10 m diameter) gate which is 10 m wide perpendicular to the paper holds water as shown in the figure. Find the force required to hold the gate. The weight of the gate can be neglected.

Solution:



Horizontal force

$$F_H = w A \bar{x}$$
$$= 9810 \times 5 \times 10 \times 5/2$$
$$= 1.226\ MN$$

And it acts at a distance of 5/3 m from the bottom end.

And the upward vertical force is the weight of the imaginary water body held over the plate.

$$F_V = 9810 \times 10 \times [5^2 - \frac{1}{4}\pi 5^2] = 5.26 \times 10^5 N$$

And it will act through its C.G. which is Determined as follow -

$$C.G.\ \text{of area}\ A_1 = \frac{4R}{3\pi} = \frac{20}{3\pi} m\ \text{away from AA}$$

C.G. of the total area about AA is 2.5m away from AA.

$$\therefore C.G.\ \text{of}\ A_2 \Rightarrow A_1 \times \frac{20}{3\pi} + A_2 \times x = A \times 2.5$$
$$x = 3.88m$$

Distance from 'O'= (5-3.88) = 1.12m.

Taking moments about $O \Rightarrow F_H \times \frac{5}{3} + F_V \times 1.12 = F \times 5$

Answer:- $5.26 \times 10^5 N$

Fluid Kinematics

- **Fluid Kinematics** deals with the motion of fluids such as displacement, velocity, acceleration, and other aspects. This topic is useful in terms of exam and knowledge of the candidate.
- Kinematics is the branch of classical mechanics that describes the motion of bodies and systems without consideration of the forces the cause the motion.

Types of Fluid Flows

Fluid flow may be classified under the following headings;

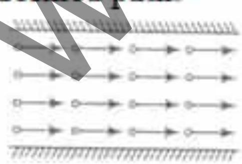
Steady & Unsteady Flow

Steady flow	**Unsteady flow**
The flow in which characteristics of fluid like velocity, pressure, density etc., at a point, do not change with time is called as steady flow. $\frac{\partial v}{\partial t}=0, \frac{\partial p}{\partial t}=0, \frac{\partial p}{\partial t}=0$	If velocity pressure and density changes with time then flow is unsteady flow. $\frac{\partial v}{\partial t}\neq 0, \frac{\partial p}{\partial t}\neq 0, \frac{\partial p}{\partial t}\neq 0$

Uniform & Non-uniform Flow

Uniform Flow	**Non-uniform Flow**
The flow in which velocity at any given time does not change with respect to distance. $\left(\frac{\partial v}{\partial s}\right)_{t=c}=0$	In this flow, velocity at any given time changes with respect to distance. $\left(\frac{\partial v}{\partial s}\right)_{t=c}\neq 0$

Laminar & Turbulent Flow

Laminar Flow	**Turbulent Flow**
The flow in which the adjacent layers do not cross each other and move along well defined path. 	The flow in which adjacent layers cross each other and do not move along well defined path. 

Rotational & Irrotational Flow

Rotational Flow	Irrotational Flow
If the fluid particles flowing alone stream lines, also rotate about their own axes, then flow is rotational.	If fluid particles do not rotate about their own axes, then flow is irrotational.

Combining these, the most common flow types are:

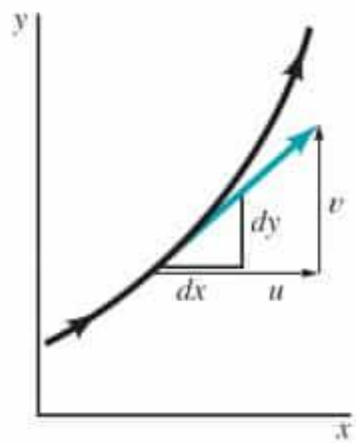
- **Steady uniform flow**
  - Conditions do not change with position in the stream or with time.
  - E.g. flow of water in a pipe of constant diameter at a constant velocity.
- **Steady non-uniform flow**
  - Conditions change from point to point in the stream but do not change with time.
  - E.g. Flow in a tapering pipe with constant velocity at the inlet.
- **Unsteady uniform flow**
  - At a given instant in time the conditions at every point are the same but will change with time.
  - E.g. A pipe of constant diameter connected to a pump pumping at a constant rate which is then switched off.
- **Unsteady non-uniform flow**
  - Every condition of the flow may change from point to point and with time at every point.
  - E.g. Waves in a channel

Flow Pattern

Three types of fluid element trajectories are defined: **Streamlines, Pathlines,** and **Streaklines**.

- **Pathline** is the actual path travelled by an individual fluid particle over some time period. The pathline of a fluid element A is simply the path it takes through space as a function of time. An example of a pathline is the trajectory taken by one puff of smoke which is carried by the steady or unsteady wind.
- **Timeline** is a set of fluid particles that form a line at a given instant.

- **Streamline** is a line that is everywhere tangent to the velocity field. Streamlines are obtained analytically by integrating the equations defining lines tangent to the velocity field as illustrated in the figure below:

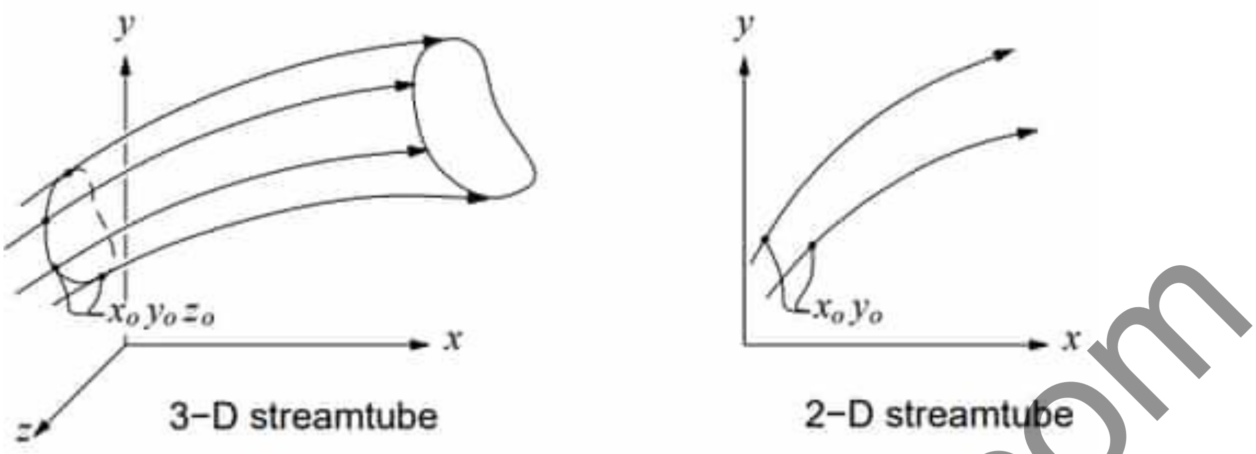


Figure

$$\frac{dy}{dx} = \frac{v}{u}$$

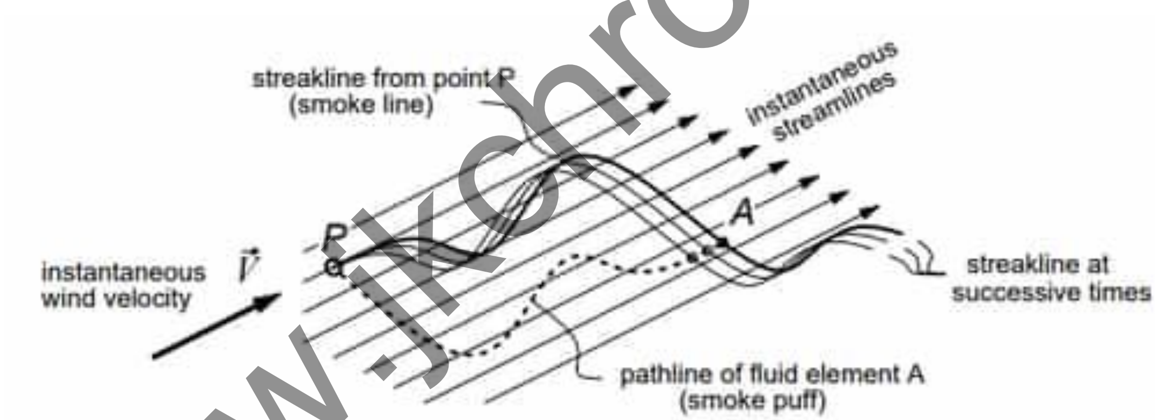
where u,v, and w are the velocity components in x, y and z directions respectively as sketched

- **Streakline** is the locus of particles that have earlier passed through a prescribed point.A streakline is associated with a particular point P in space which has the fluid moving past it. All points which pass through this point are said to form the streakline of point P. An example of a streakline is the continuous line of smoke emitted by a chimney at point P, which will have some curved shape if the wind has a time-varying direction
- **Streamtube**: The streamlines passing through all these points form the surface of a stream-tube. Because there is no flow across the surface, each cross-section of the streamtube carries the same mass flow. So the streamtube is equivalent to a channel flow embedded in the rest of the flow field.



Note:

- The figure below illustrates **streamlines, pathlines**, and **streaklines** for the case of a smoke being continuously emitted by a chimney at point P, in the presence of a shifting wind.
- In a steady flow, streamlines, pathlines, and streaklines all coincide.
- In this example, they would all be marked by the smoke line.



Velocity of Fluid Particle

- Velocity of a fluid along any direction can be defined as the rate of change of displacement of the fluid along that direction
- Let **V** be the resultant velocity of a fluid along any direction and ***u***, **v** and **w** be the velocity components in ***x, y*** and ***z*** directions respectively.
- Mathematically the velocity components can be written as

$$u = f(x, y, z, t)$$

w = f (x, y, z, t)

v = f (x, y, z, t)

- Let V_R is resultant velocity at any point in a fluid flow.
- Resultant velocity VR **=** u**i +** v**j +** w**k**

$$V_R = \sqrt{u^2 + v^2 + w^2}$$

Where **u=dx/dt, v=dy/**dt **and w=dz/dt** are the resultant vectors in X, Y and Z directions, respectively.

Acceleration of Fluid Particle

- Acceleration of a fluid element along any direction can be defined as the rate of change of velocity of the fluid along that direction.
- If a_x , a_y and a_z are the components of acceleration along ***x, y*** and ***z*** directions respectively, they can be mathematically written as $a_x = du/dt$.

Similarly

$$a_y = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} + \frac{\partial v}{\partial z}\frac{dz}{dt} + \frac{\partial v}{\partial t}$$

and

$$a_z = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial t}$$

But $u = (dx/dt)$, $v = (dy/dt)$ and $w = (dz/dt)$.

Hence

$$\left.\begin{aligned} a_x &= \overbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}^{\text{Convective acceln}} + \overbrace{\frac{\partial u}{\partial t}}^{\text{Local acceln}} \\ a_y &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\ a_z &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \end{aligned}\right\} \text{Total acceln}$$

If A is the resultant acceleration vector, it is given by

$$A = a_x i + a_y j + a_z k$$
$$= \sqrt{a_x^2 + a_y^2 + a_z^2}$$

For steady flow, the local acceleration will be zero

Stream Function

- The partial derivative of stream function with respect to any direction gives the velocity component at right angles to that direction. It is denoted by **ψ**.

$$\frac{\partial \psi}{\partial x} = v, \frac{\partial \psi}{\partial \psi} = -u$$

- Continuity equation for two-dimensional flow is

$$-\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Equations of Rotational Flow

- As ψ satisfies the continuity equation hence if ψ exists then it is a possible case of fluid flow.
- Rotational components of fluid particles are:

$$\omega_x = \frac{1}{2}\left(\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right)$$

Equation of Irrotational Flow

- If $\omega_x = \omega_y = \omega_z$ then, flow is irrotational.
- For irrotational flow, $\omega_z = 0$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- This is **Laplace equation** for ψ.

Note: It can be concluded that if stream function (ψ) exits, it is a possible case of fluid flow. But we can't decide whether flow is rotational or irrotational. But if stream function ψ satisfies Laplace equation then, it is a possible case of irrotational flow otherwise it is rotational flow.

Velocity Potential Function

- It is a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is denoted b φ

$$-\frac{\partial \phi}{\partial x} = u, -\frac{\partial \phi}{\partial y} = v, -\frac{\partial \phi}{\partial z} = w,$$

We know that continuity equation for steady flow is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right)=0$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0$$

- If φ satisfies the Laplace equation, then it is a possible case of fluid flow.

Rotational component *ω_z* can be given by

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{-\partial \phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right]$$
$$=\frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x\cdot\partial y}+\frac{\partial^2 \phi}{\partial y\cdot\partial x}\right)=0$$

- It shows that φ exits then, flow will be irrotational.

Relation between Stream Function and Velocity Potential

We know,

$$-\frac{\partial \phi}{\partial x}=u,-\frac{\partial \phi}{\partial y}=v$$

and

$$\frac{\partial \psi}{\partial x}=v,-\frac{\partial \psi}{\partial y}=u$$

$$\therefore\ \frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y}\ and\ \frac{\partial \phi}{\partial x}=-\frac{\partial \psi}{\partial x}$$

Stream versus Velocity Function

Stream Function	Velocity Potential
$\frac{\partial \psi}{\partial x} = v$ $\frac{\partial \psi}{\partial y} = -u$ If Ψ exists then, a possible case of fluid flow. If Ψ satisfies Laplace equation then, flow is irrotational.	$-\frac{\partial \phi}{\partial x} = u$ $-\frac{\partial \phi}{\partial y} = v$ If φ exists then, flow is irrotational. If φ satisfies the Laplace equation then, a possible case of fluid flow.

Equipotential Line versus Stream Line

Equipotential Line	Stream Line
It is the line along which velocity potential φ is constant. $d\phi = \frac{d\phi}{dx} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy = 0$ $\Rightarrow \; -u.dx - v.dy = 0$ $\Rightarrow \; \frac{dy}{dx} = \frac{-u}{v}$	It is the line along which stream function Ψ is constant. $d\psi = \frac{\partial \psi}{\partial x} \cdot dx + \frac{\partial \psi}{\partial y} \cdot dy = 0$ $\Rightarrow \; -v\,dx - u\,dy = 0$ $\Rightarrow \; \frac{dy}{dx} = \frac{v}{u}$

Fluid Dynamics and Flow Measurements

Fluid Dynamics

Fluid Dynamics is the beginning of the determination forces which cause motion in fluids. This section includes various forces such as Inertia, Viscous, etc., Bernoulli's theorems, Vortex motion, forced motion etc.

Include momentum correction factor, the impact of jets etc.

Dynamics is that branch of mechanics which treats the motion of bodies and the action of forces in producing or changing their motion.

Flow rate

- **Mass flow rate**

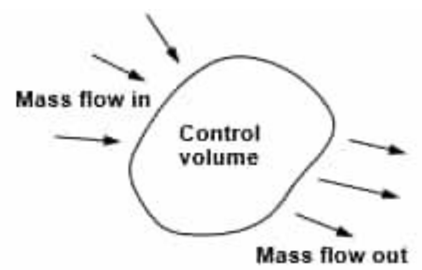
$$\dot{m} = \frac{dm}{dt} = \frac{mass}{time\ taken\ to\ accumalate\ mass}$$

- **Volume flow rate - Discharge**
    - More commonly we use volume flow rate Also know as discharge. The symbol normally used for discharge is Q.

Discharge, Q = Volume / Time

Continuity

This principle of conservation of mass says matter cannot be created or destroyed. This is applied in fluids to fixed volumes, known as control volumes (or surfaces).



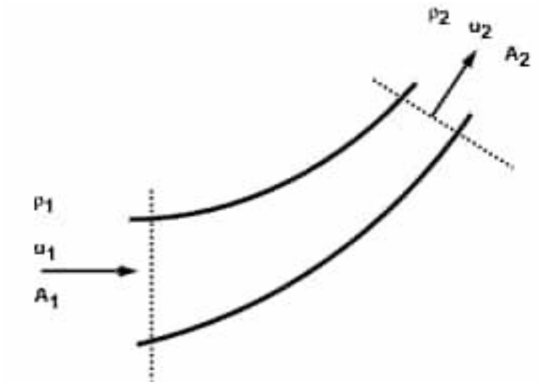
- For any control volume, the principle of conservation of mass defines,

Mass entering per unit time = Mass leaving per unit time + Increase of mass in control vol per unit time

- For steady flow there is no increase in the mass within the control volume,

Mass entering per unit time = Mass leaving per unit time

Applying to a stream-tube
Mass enters and leaves only through the two ends (it cannot cross the stream tube wall).

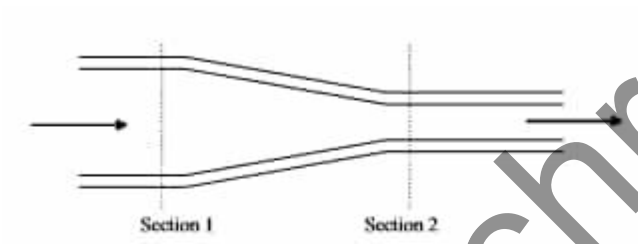


for steady flow,

$\rho_1 \partial A_1 u_1 = \rho_2 \partial A_2 u_2$= Constant= Mass flow rate

This is the continuity equation.

Some example applications of Continuity

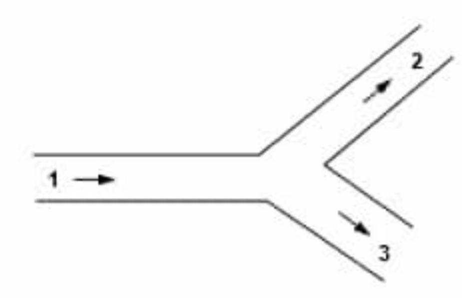


A liquid is flowing from left to right. By the continuity, **$\rho_1 A_1 u_1 = \rho_2 A_2 u_2$**

As we are considering a liquid,

$\rho_1=\rho_2$
$Q_1 = Q_2$
$A_1u_1=A_2u_2$

Velocities in pipes coming from a junction



mass flow into the junction = mass flow out

$\rho_1 Q_1 = \rho_2 Q_2 + \rho_3 Q_3$

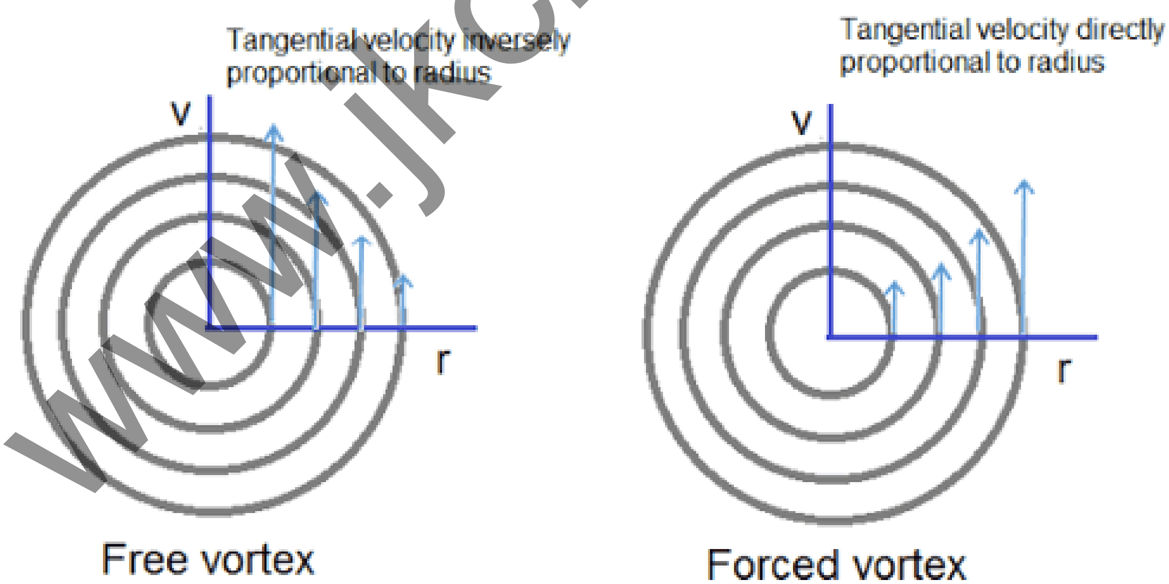
When incompressible,

$Q_1 = Q_2 + Q_3$

$A_1 u_1 = A_2 u_2 + A_3 u_3$

Vortex flow

- This is the flow of rotating mass of fluid or flow of fluid along curved path.



Free vortex flow

- No external torque or energy required. The fluid rotating under certain energy previously given to them. In a free vortex mechanics, overall energy flow remains constant. There is no energy interaction between an external source and a flow or any dissipation of mechanical energy in the flow.
- Fluid mass rotates due to the conservation of angular momentum.
- Velocity inversely proportional to the radius.
- For a free vortex flow

vr= constant

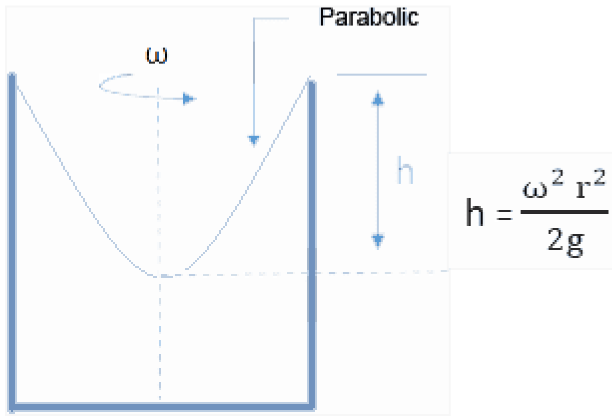
v= c/r

- **At the center (r = 0) of rotation, velocity approaches to infinite, that point is called singular point.**
- The free vortex flow is **irrotational**, and therefore, also known as the irrotational vortex.
- In free vortex flow, Bernoulli's equation can be applied.

***Examples* include a whirlpool in a river, water flows out of a bathtub or a sink, flow in centrifugal pump casing and flow around the circular bend in a pipe.**

Forced vortex flow

- To maintain a forced vortex flow, it required a continuous supply of energy or external torque.
- All fluid particles rotate at the constant angular velocity ω as a solid body. Therefore, a flow of forced vortex is called as a solid body rotation.
- Tangential velocity is directly proportional to the radius.
    - v = r ω
    - ω = Angular velocity.
    - r = Radius of fluid particle from the axis of rotation.
- The surface profile of vortex flow is parabolic.



- In forced vortex total energy per unit weight increases with an increase in radius.
- Forced vortex is not irrotational; rather it is a **rotational flow** with constant vorticity 2ω.

Examples of forced vortex flow is rotating a vessel containing a liquid with constant angular velocity, flow inside the centrifugal pump.

Energy Equations

- This is the equation of motion in which the forces due to gravity and pressure are taken into consideration. The common fluid mechanics equations used in fluid dynamics are given below
- Let, **Gravity force F_g, Pressure force F_p, Viscous force** F_v , **Compressibility force** F_c , and **Turbulent force F_t**.

$$F_{net} = F_g + F_p + F_v + F_c + F_t$$

- If fluid is **incompressible**, then $F_c = 0$

$$\therefore F_{net} = F_g + F_p + F_v + F_t$$

This is known as **Reynolds equation of motion**.

- If fluid is **incompressible and turbulence** is negligible, then, $F_c = 0, F_t = 0$

$$\therefore F_{net} = F_g + F_p + F_v$$

This equation is called as **Navier-Stokes equation**.

- If fluid flow is considered **ideal** then, a **viscous effect** will also be negligible. Then

$F_{net} = F_g + F_p$

This equation is known as **Euler's equation**.

- Euler's equation can be written as:

$$\frac{dp}{\rho} + gdz + vdv = 0$$

Bernoulli's Equation

It is based on law of conservation of energy. This equation is applicable when it is assumed that

- Flow is steady and irrotational
- Fluid is ideal (non-viscous)
- Fluid is incompressible

It states in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant.

The total energy consists of pressure energy, kinetic energy and potential energy or datum energy. These energies per unit weight of the fluid are:

- Pressure energy

$$= \frac{p}{\rho g}$$

- Kinetic energy

$$= \frac{v^2}{2g}$$

- Datum energy = z

Bernoulli's theorem is written as:

$$\frac{p}{w} + \frac{v^2}{2g} + z = constant$$

- Bernoulli's equation can be obtained by **Euler's equation**

$$\frac{dp}{\rho} + v\,dv + g\,dz = constant$$

As fluid is incompressible, ρ = constant

$$\int \frac{dp}{\rho} = \int v\,dv + \int g\,dz + \text{constant}$$

$$or\ \ \underbrace{\frac{\rho}{\rho g}}_{\text{Pressure head}} + \underbrace{\frac{v^2}{2g}}_{\text{Kinetic head}} + \underbrace{z}_{\text{Potential head}} = \text{constant}$$

where, Head = Energy / Weight

- **Restrictions inthe application of Bernoulli's equation**
  - Flow is steady
  - Density is constant (incompressible)
  - Friction losses are negligible
  - It relates the states at two points along a single streamline, (not conditions on two different streamlines)

The Bernoulli equation is applied along streamlines like that joining points 1 and 2

Total head at 1 = Total head at 2

$$\frac{p_1}{\rho g}+\frac{u_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{u_2^2}{2g}+z_2$$

This equation assumes no energy losses (e.g. from friction) or energy gains (e.g. from a pump) along the streamline. It can be expanded to include these simply, by adding the appropriate energy terms

Total energy per unit weight at 1	=	Total energy per unit weight at 2	+	Loss per unit weight	+	Work done per unit weight	−	Energy supplied per unit weight

$$\frac{p_1}{\rho g}+\frac{u_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{u_2^2}{2g}+z_2+h+w-q$$

Note Point:

The Bernoulli equation is often combined with the continuity equation to find velocities and pressures at points in the flow connected by a streamline.

Kinetic Energy Correction Factor (α)

In a real fluid flowing through a pipe or over a solid surface, the velocity will be zero at the solid boundary and will increase as the distance from the boundary increases. The kinetic energy per unit weight of the fluid will increase in a similar manner.

The kinetic energy in terms of **average velocity V** at the section and a kinetic energy correction factor α can be determined as:

$$\text{K.E.}=\alpha\frac{V^2}{2}m=\alpha\frac{\rho dt}{2}AV^3$$

In which m = ρAVdt is the total mass of the fluid flowing across the cross-section during dt. By comparing the two expressions for kinetic energy, it is obvious that,

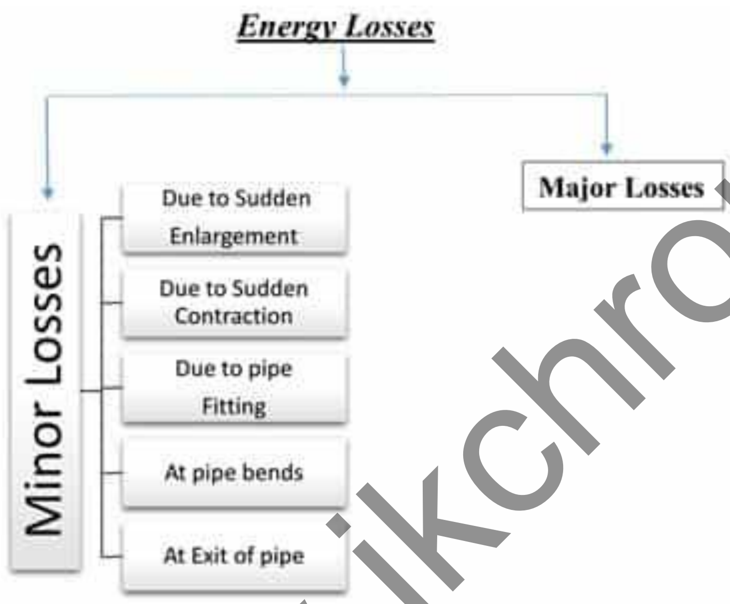
$$\alpha = \frac{1}{AV^3}\int_A u^3 dA$$

The numerical value of α will always be greater than 1

Flow-through Pipes & Boundary Layer Theory

This article contains basic notes on "**Flow-through Pipes**" topic of "**Fluid Mechanics & Hydraulics**" subject.

Flow-through Pipes



Major Loss: It is calculated by Darcy Weisbach formulas

Loss of head due to friction

$$h_f = \frac{4fLv^2}{d \cdot 2g}$$

where,

L = Length of pipe,

v = Mean velocity of flow

d = Diameter of pipe,

f = Coefficient of friction

$$or\ h_f = \frac{f'Lv^2}{d2g}\ f' = 4f =$$ friction factor

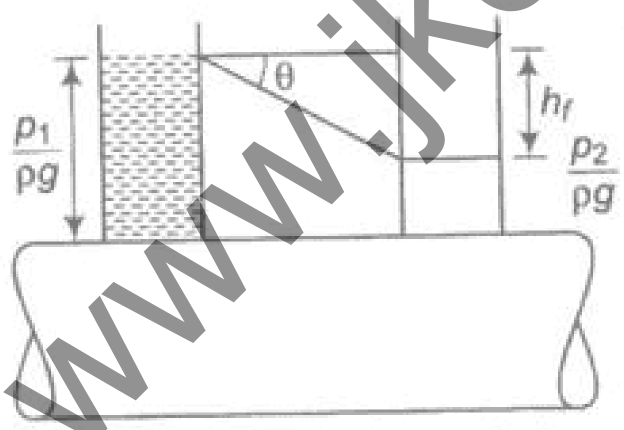
$$\frac{\text{for laminar flow}}{\text{frictional factor(4f)}}\ f' = \frac{64}{\text{Re}}$$

$$\text{Coefficient of friction } f = \frac{16}{\text{Re}}$$

For turbulent flow, coefficient of friction

$$f = \frac{0.079}{\text{Re}^{\frac{1}{4}}}$$

Chezy's Formula: In fluid dynamics, Chezy's formula describes the mean flow velocity of steady, turbulent open channel flow.



Chezy's formula of steady flow

v= c √mi, **c= Chezy's Constant = √(8g/f)**

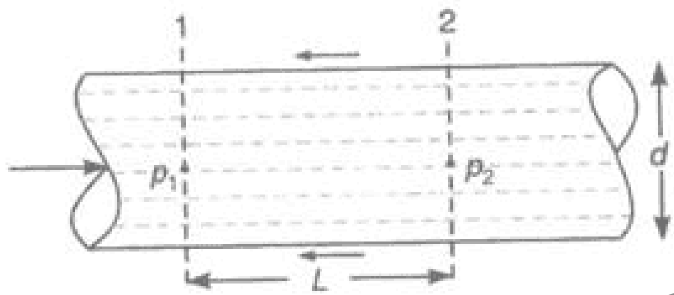
i = Loss of head per unit length of pipe

$= \frac{h_f}{L}$ (hydraulic slope tan θ)

m = Hydraulic mean depth

$$= \frac{Area(A)}{Wetted\ \ perimetr\,(p)}$$

Relation between Coefficient of Friction and Shear Stress



Coefficient of friction and shear stress

We get $f = \frac{2\tau_0}{\rho v^2}$

where,

f = Coefficient of friction

τ_0 = Shear stress

Minor Loss:

The Another type of head loss in minor loss is induced due to following reasons

Loss due to Sudden Enlargement

Head loss

$$h_L = \frac{(v_1 - v_2)^2}{2g}$$

Loss due to Sudden Contraction

Head loss, $h_L = 0.5\, v_2^2 / 2g$

Remember v_2 is velocity at point which lies in contracted section.

Loss of Head at Entrance to Pipe

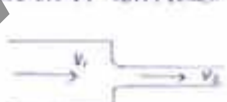
Head loss,

$$h_L = 0.5 \frac{v_1^2}{2g}$$

Loss at Exit from Pipe

Head loss,

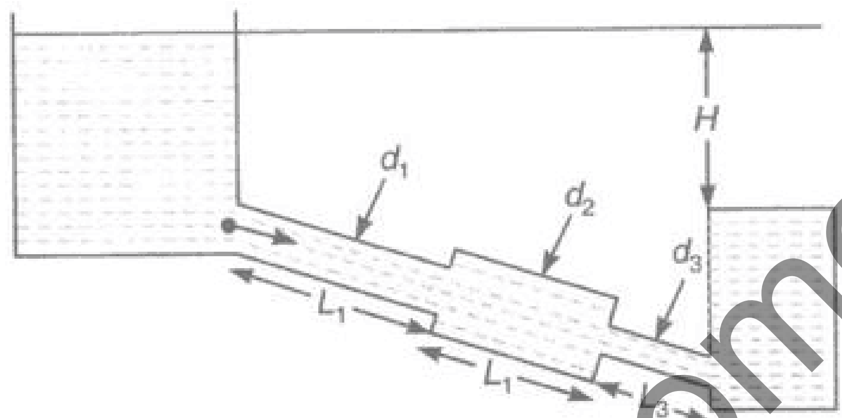
$$h_L = \frac{v_1^2}{2g}$$

Expression	***Expression***
Sudden expansion 	$H_L = \frac{(V_1 - V_2)^2}{2g}$
Exit from pipe 	$H_L = \frac{V^2}{2g}$
Sudden Contraction 	$H_L = \left(\frac{1}{C_c} - 1\right)^1 \frac{V_2^2}{2g} \approx 0.5 \frac{V_2^2}{2g}$ C_c = Coefficient of contraction $C_c = \frac{A_c}{A_2}$ and $A_c \rightarrow$ Cross section Area at vana contracta
Head loss at pipe bend or head loss due to pipe fitting	$H_L = \frac{kV^2}{2g}$

Note: In case 1 and 2, flow occurs between pipe to pipe, while in case 3 and 4, flow occurs between tank and pipe. We are taking entry or exit w.r.t. pipe. So, be careful.

Combination of Pipes: Pipes may be connected in series, parallel or in both. Let see their combinations.

Pipe in Series: As pipes are in series, the discharge through each pipe will be same.



Pipe in series

$$Q = A_1v_1 = A_2v_2 = A_3v_3$$

Total loss of head = Major loss + Minor loss

$$H = h_{L_1} + h_{L_2}$$

Major loss = Head loss

due to friction in each pipe

$$h_{L_1} = h_{f_1} + h_{f_2} + h_{f_3}$$
$$= \frac{f_1 L_1 v_1^2}{d_1.2g} + \frac{f_2 L_2 v_2^2}{d_2.2g} + \frac{f_3 L_3 v_3^2}{d_3.2g}$$

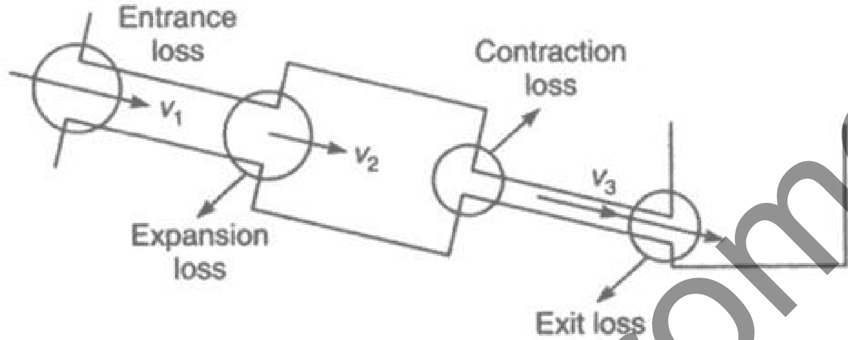
While, minor loss = Entrance loss + Expansion loss + Contraction loss + Exit loss

$$h_{L_2} = \frac{0.5v_1^2}{2g} + \frac{(v_2 + v_1)^2}{2g} + \frac{0.5v_3^2}{2g} + \frac{v_3^2}{2g}$$

If minor loss are neglected then,

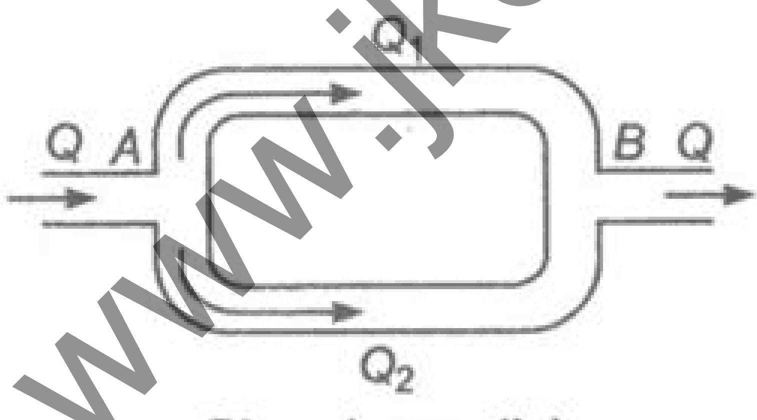
$$H = \frac{f_1 L_1 v_1^2}{d_1.2g} + \frac{f_2 L_2 v_2^2}{d_2.2g} + \frac{f_3 L_3 v_3^2}{d_3.2g}$$

$$H = \frac{f_1 L_1 Q^2}{12 \cdot 1 d_1^5} + \frac{f_2 L_2 Q^2}{12 \cdot 1 d_2^5} + \frac{f_3 L_3 Q^2}{12 \cdot 1 d_3^5}$$



Head loss when minor loss are neglected

Pipes in Parallel: In this discharge in main pipe is equal to sum of discharge in each of parallel pipes.



Pipes in parallel

Hence, Q = $Q_1 + Q_2$

Loss of head in each parallel pipe is same

$$h_{f_1} = h_{f_2}$$

$$\frac{f_1 L_1 v_1^2}{d_1 \cdot 2g} = \frac{f_2 L_2 v_2^2}{d_2 \cdot 2g} \ or \ \frac{f_1 L_1 Q_1^2}{12.1 d_1^5} = \frac{f_2 L_2 Q_2^2}{12.1 d_2^5}$$

where, h_{f_1} and h_{f_2} are head loss at 1 and 2 respectively.

Equivalent Pipe: A compound pipe which consists of several pipes of different lengths and diameters to be replaced by a pipe having uniform diameter and the same length as that of compound pipe is called as equivalent pipe.

$$h_{Le} = h_{f_1} + h_{f_2} + h_{f3}$$

$$\frac{fLQ^2}{12.1d^5} = \frac{f_1 L_1 Q^2}{12.1 d_1^5} + \frac{f_2 L_2 Q^2}{12.1 d_2^5} + \frac{f_3 L_3 Q^2}{12.1 d_3^5}$$

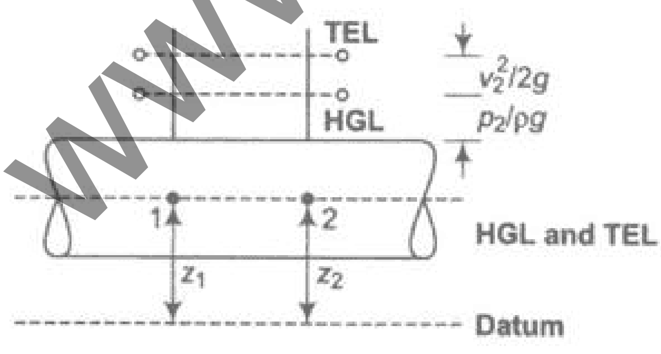
(where, $L = L_1 + L_2 + L_3$)

If $f = f_1 = f_2 = f_3$

Then,

$$\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5} \Rightarrow \frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$$

Hydraulic Gradient Line (HGL) and Total Energy Line (TEL)



Equivalent pipe diagram

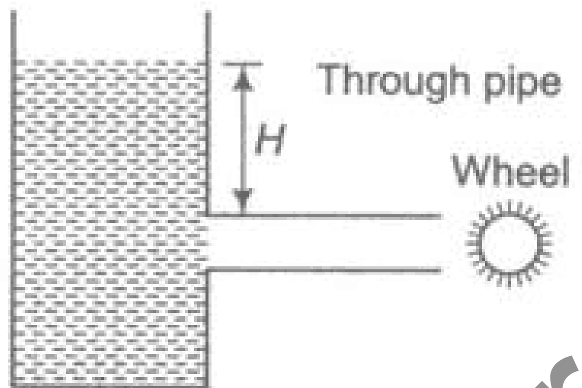
HGL → It joins piezometric head (p/ρg + z) at various points.

TEL → It joins total energy head at various points:

{(p/ρg + z) + v²/2g}

Note: HGL is always parallel but lower than TEL.

Power Transmission through Pipe (P)



Power transmission through pipe

$$P_{ideal} = \rho g Q H$$
$$P_{actual} = \rho g Q(H - h_f)$$
$$h_f = Head\, loss$$
$$Efficiency\, \eta = \frac{H - h_f}{H}$$

Power delivered by a given pipe line is maximum when the flow is such that one third of static head is consumed in pipe friction. Thus, efficiency is limited to only 66.66%

Maximum efficiency,

$$\eta_{max} = \frac{H}{3}$$

Water Hammer: When a liquid is flowing through a long pipe fitted with a vale at the end of the pipe and the valve is closed suddenly a pressure wave of high intensity is produced behind the valve. This pressure wave of high intensity is having the effect of hammering action on the walls of the pipe. This phenomenon is known as **water hammer**.

Intensity of pressure rise due to water hammer,

$$\rho = -\frac{\rho L v}{t}$$

When valve is closed gradually when valve closed suddenly with rigid pipe.

$$\rho = v\sqrt{K\rho}$$

When valve closed suddenly with plastic pipe

$$\rho = v \times \sqrt{\frac{\rho}{\frac{1}{K} + \frac{D}{Et}}}$$

If the time required to close the valve

$t > \frac{2L}{C} \Rightarrow$ Valve closure is said to be gradual.

$t < \frac{2L}{C} \Rightarrow$ The valve closure is said to be sudden.

Where,

L = Length of pipe

D = Diameter of pipe

C = Velocity of pressure wave produced due to water hammer

$$\sqrt{\frac{K}{\rho}}$$

v = Velocity of flow

K = Bulk modulus of water

E = Modulus of elasticity for pipe material.

t = Time required to choose the valve.

Boundary-Layer Theory

Boundary Layer Theory

When a real fluid flows over a solid body, the velocity of fluid at the boundary will be zero. If boundary is stationary. As we move away from boundary in perpendicular direction velocity increases to the free stream velocity. It means velocity gradient $\left(\frac{du}{dy}\right)$ will exist.

Note:

Velocity gradient $\frac{du}{dy}$ does not exist outside the boundary layer as outside the boundary layer velocity is constant and equal to free stream velocity.

Development of Boundary Layer: Development of boundary layer can be divided in three regions: laminar, transition, turbulent.

Reynolds number

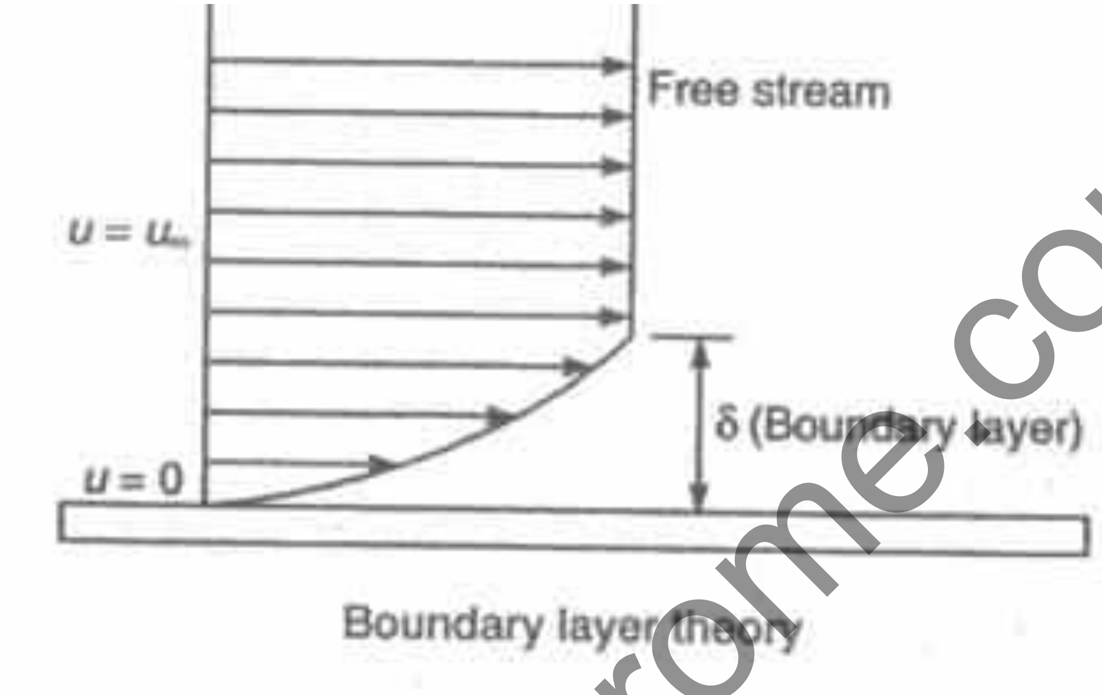
$$= \frac{\rho v \cdot x}{\mu} (\text{Re})_x = \frac{v \cdot x}{\nu}$$

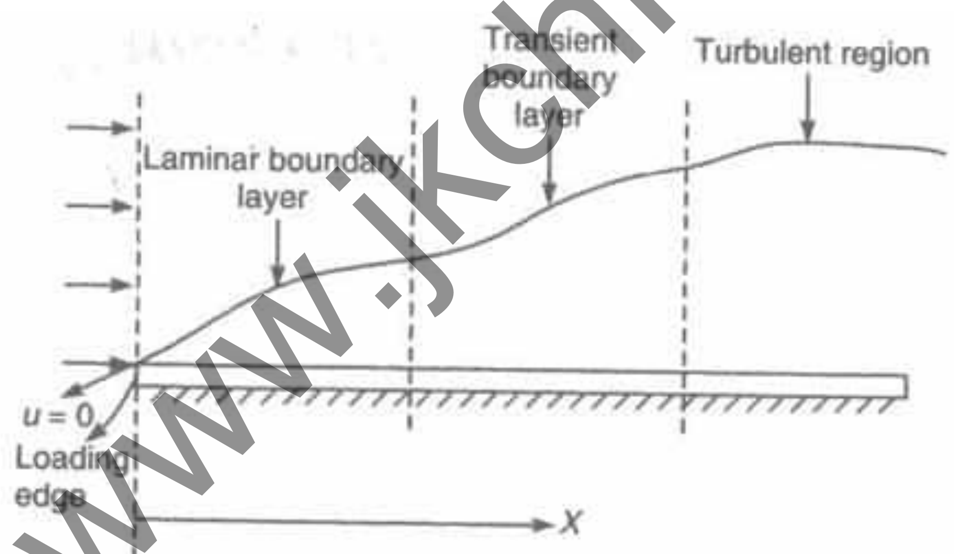
For **laminar** boundary layer

$(\text{Re})_x <$ **5×10^5** (For flat plate) and if $(\text{Re})_x >$ **5×10^5**

where R_e = Reynolds's number

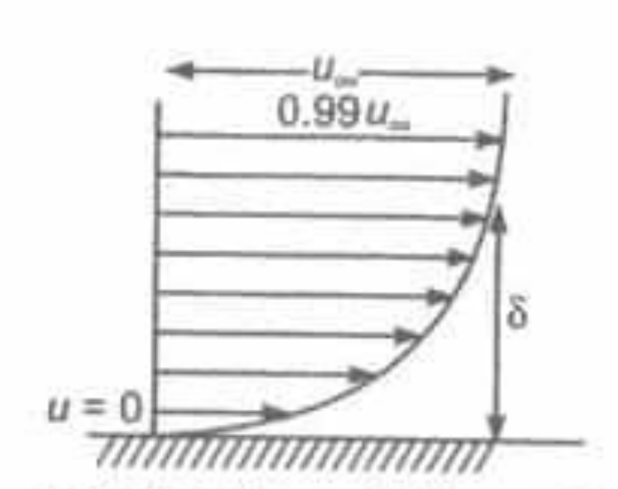
Then, flow is **turbulent**.





Here, x is distance from leading edge in horizontal direction.

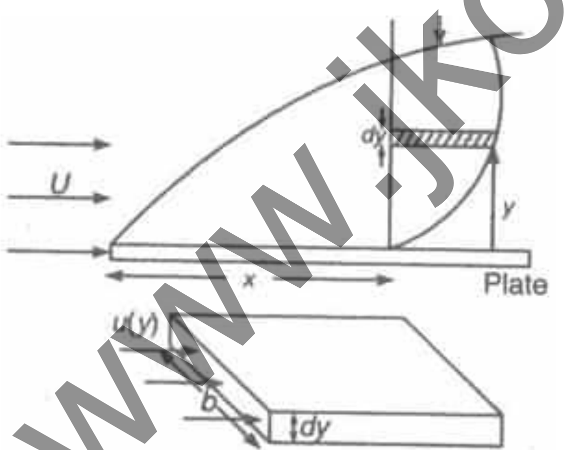
Boundary Layer Thickness (δ): It is the distance from the boundary to the point where velocity of fluid is approximately equal to 99% of free stream velocity. It is represented by δ.



Boundary Layer thickness δ

Displacement Thickness (δ*): It is observed that inside the boundary layer velocity of fluid is less than free stream velocity hence, discharge is less in this region. To compensate for reduction in discharge the boundary is displaced outward in perpendicular direction by some distance. This distance is called displacement thickness (δ*).

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$



Boundary layer displacement thickness (δ*)

Momentum Thickness (θ): As due to boundary layer reduction in velocity occurs so, momentum also decreases. Momentum thickness is defined as the distance measured normal to boundary of solid body by which the boundary should be displaced to compensate for the reduction in momentum of flowing fluid.

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

Energy Thickness (δ):** It is defined as distance measured perpendicular to the boundary of solid body by which the boundary should be displaced to compensate for reduction in kinetic energy of flowing fluid (KE decreases due to formation of boundary layer)

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy$$

Boundary Conditions for the Velocity Profile: Boundary conditions are as

$$(a)\ At\ y=0, u=0, \frac{du}{dy}\neq 0$$

$$(b)\ At\ y=\delta, u=U, \frac{du}{dy}=0$$

Laminar Flow: A flow in which fluid flows in layer and no intermixing with each other is known as laminar flow. For circular pipe, flow will be laminar.

$$\text{If Re} = \frac{\rho v D}{\mu} < 2000$$

Where, ρ = Density of fluid, v = Velocity of fluid,

D = Diameter of pipe, μ = Viscosity of fluid.

For flat plate flow will be laminar.

$$\text{If Re} = \frac{\rho v L}{\mu} < 5\times 10^5$$

Where L is length of plate.

Turbulent Flow:

In this flow, adjacent layer of fluid cross each other (particles of fluid move randomly instead of moving in stream line path), for flow inside pipe. If Re > 4000, the flow is considered turbulent, for flat plate, Re > 5×10^5.

Von Karman Momentum Integral Equation

$$\frac{\tau_0}{\rho U^2} = \frac{d\theta}{dx}$$

where, θ = momentum thickness

Shear stress:

$$\tau_0 = \mu \left(\frac{du}{dy} \right)_{y=0}$$

Where, U = Free stream velocity; ρ = Density of fluid.

Local Coefficient of Drag (C_D*):

It is defined as the ratio of the shear stress τ_0 to the quantity

$$\frac{1}{2} \rho U^2.$$

It is denoted by

$$C_D^* = \frac{\tau_0}{\frac{1}{2} \rho U^2}$$

Average Coefficient of Drag (C_D):

It is defined as the ratio of the total drag force to

$$\frac{1}{2} \rho A U^2$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}$$

Where, A = Area of surface,

U = Free stream velocity, ρ = Mass density of fluid.

Blassius Experiment Results

For laminar flow,

$$\frac{f}{x} = \frac{5.48}{\sqrt{\mathrm{Re}_x}}$$

Coefficient of drag

$$C_{fx} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

Average coefficient of drag

$$C_D = \frac{1.46}{\sqrt{\mathrm{Re}_L}}$$

For turbulent flow,

$$\frac{f}{x} = \frac{0.37}{(\mathrm{Re}_x)^{\frac{1}{5}}},$$

where x = Distance from leading edge, Re_X = Reynolds's number for length x.

Re_L = Reynolds's number at end of plate

Coefficient of drag

$$C_{fx} = \frac{0.059}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}}$$

Average coefficient of drag

$$C_D = \frac{0.072}{\left(\mathrm{Re}_L\right)^{\frac{1}{5}}}$$

For laminar flow

$$f \propto \sqrt{x}$$

f = Boundary layer thickness,

$$\tau_0 \propto \frac{1}{\sqrt{x}}$$

τ_0 = Shear stress at solid surface

x = Distance from where solid surface starts.

Velocity profile for turbulent boundary layer is

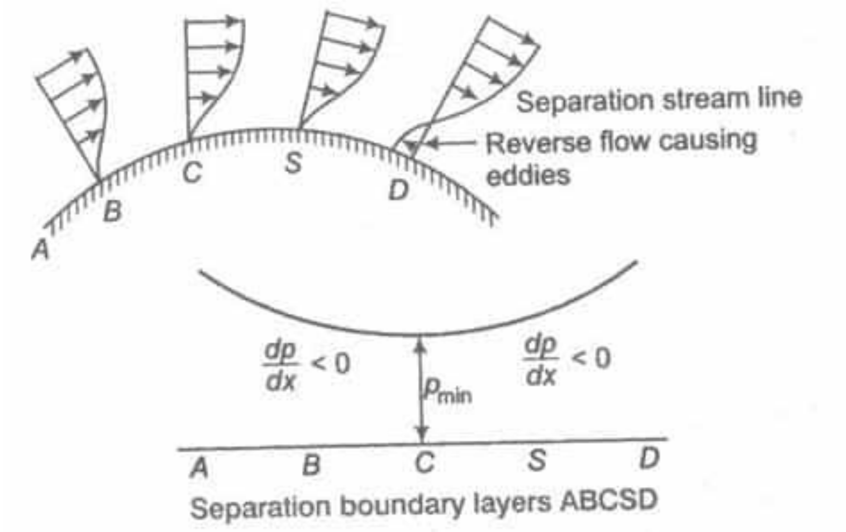
$$\frac{u}{U} = \left(\frac{y}{f}\right)^{\frac{1}{n}} \Rightarrow \frac{1}{n} = \frac{1}{\tau}$$

$$\Rightarrow 5\times10^5 < \mathrm{Re} < 107$$

Conditions for Boundary Layer Separation: Let us take curve surface ABCSD where fluid flow separation print S is determined from the condition

$$\left(\frac{du}{dy}\right)_y = 0$$

.



If

$$\left(\frac{du}{dy}\right)_{y=0} = -ve,$$

the flow is separated

$$\left(\frac{dp}{dx} > 0\right)$$

If

$$\left(\frac{du}{dy}\right)_{y=0} = 0,$$

the flow is on the average of separation

$$\left(\frac{\partial p}{\partial x} = 0\right)$$

If

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = +ve,$$

the flow will not separate or flow will remain attained

$$\left(\frac{dp}{dx} < 0\right).$$

Methods of Preventing Separation of Boundary Layer: Suction of slow-moving fluid by a suction slot.

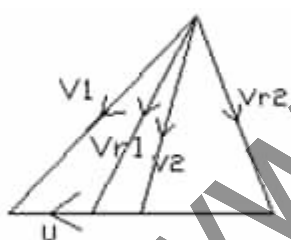
- Supplying additional energy from a blower.
- Providing a bypass in the slotted
- Rotating boundary in the direction of flow.
- Providing small divergence in a diffuser.
- Providing guide blades in a bend.
- Providing a trip wire ring in the laminar region for the flow over a sphere.

Pumps and Turbines

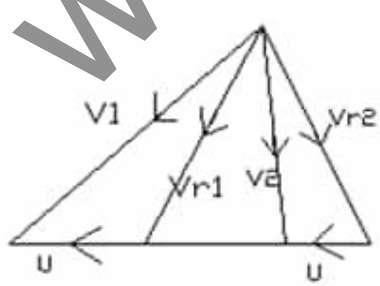
Velocity Diagrams

Velocity diagrams for different values of Rd appear as shown below:

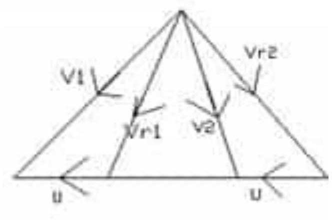
(i) $V_1 > V_2$, $V_{r1} > V_{r2}$; $R_d < 0$ (R_d is negative)



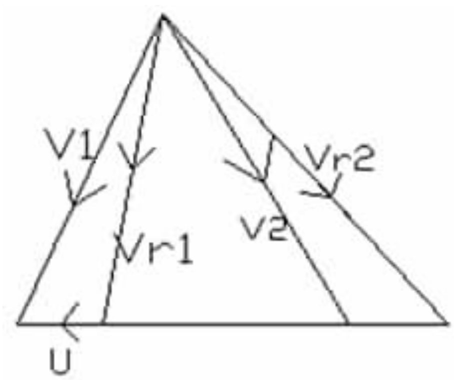
(ii) $V_{r1} = V_{r2}$; $R_d = 0$



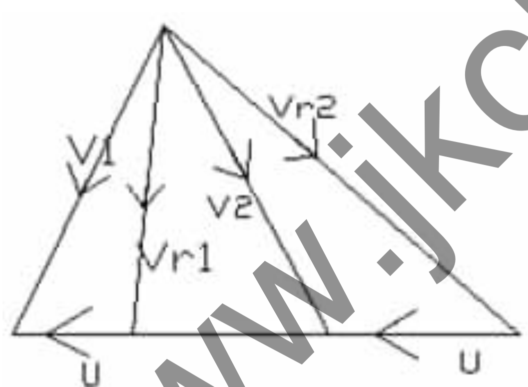
(iii) $V_1=V_{r2}$; $V_{r1}=V_2$; $R_d=0.5$



(iv) $V_1=V_2$; $R_d=100\%$

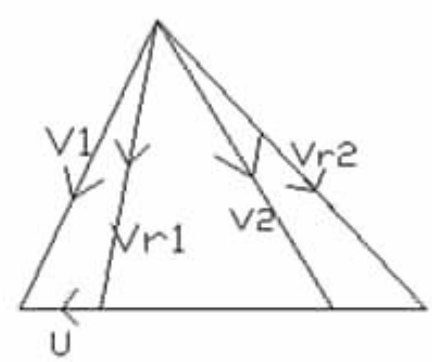


(v) $V_2>V_1$; $V_{r2}>V_{r1}$; $R_d>100\%$



We know that for utilization factor ε to be maximum, the exit velocity V2 should be minimum.

For a given rotor speed U, the minimum value of V2 is obtained only if V22 is axial and the velocity triangles would look as shown:



Velocity triangle for maximum utilization factor condition:

$$V_2 = V_1 \sin\alpha_1$$

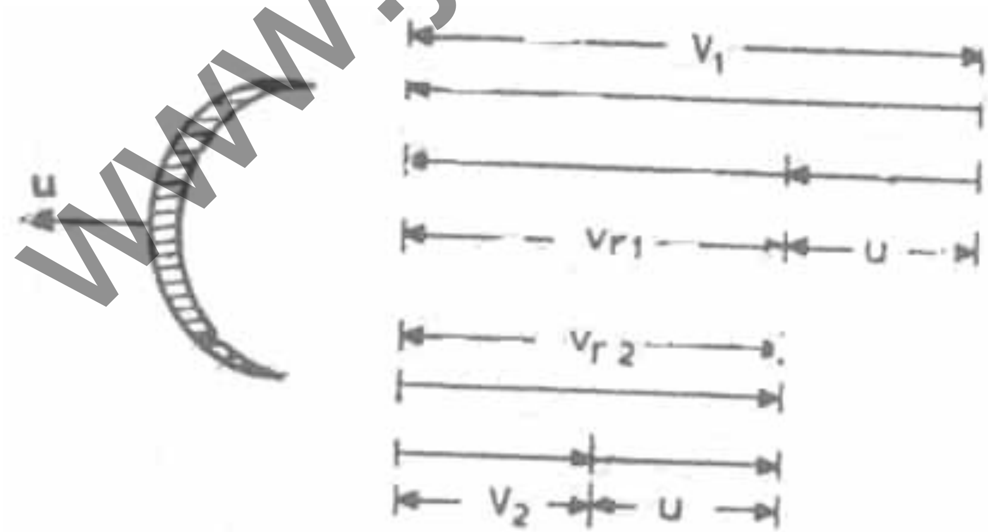
$$\varepsilon = \frac{V_1^2 - V_2^2}{V_1^2 - R_d V_2^2}$$

$$\varepsilon_{maximum} = \frac{V_1^2 - V_1^2 \sin^2\alpha_1}{V_1^2 - R_d V_1^2 \sin^2\alpha_1}$$

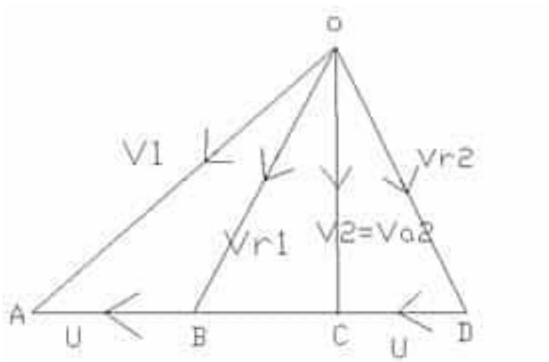
$$\varepsilon_{maximum} = \frac{\cos^2\alpha_1}{1 - R_d \sin^2\alpha_1}$$

Form the expression, it is clear that $\varepsilon_{maximum}$ will have the highest value if α_1 = 0.

But α_1 = 0, results in V2 = 0 which is not a practically feasible condition. The zero angle turbines which would have α_1 = 0 appears as shown:



Though the zero angle turbines are not practically feasible it represents the ideal condition to be aimed at. In a Pelton wheel, we arrive at a condition wherein the jet is deflected through an angle of 165 to 170 degrees. Though an angle of 180 degrees would be the ideal condition as in case of a zero angle turbine. Impulse turbine designed for maximum utilization.



$$\cos\alpha_1 = \frac{AC}{OA} = \frac{2U}{V_1}$$

$$\frac{\cos\alpha_1}{2} = \frac{U}{V_1}$$

The ratio $\frac{U}{V_1}$ is referred to as a blade speed ratio φ which will have limiting value of 0.5 for a zero angle turbine. But in practical situation, α_1 is in between 20 to 25 degrees. But φ varies from 0.45 to 0.47. The blade speed ratio is very useful performance parameter and it may be noted that the closer its value is to 0.5, the better it is.

Expression for power output:

$$P = (V_{w1}U_1 - V_{w2}U_2) \quad \frac{Kj}{Kg}$$

$V_{w2} = 0$ for $\varepsilon_{maximum}$ condition.

$$P = V_{w1}U$$

but, $V_{w1} = V_1\cos\alpha_1 = 2U$

$$\therefore P = 2U^2$$

Reaction turbine:

We know that,

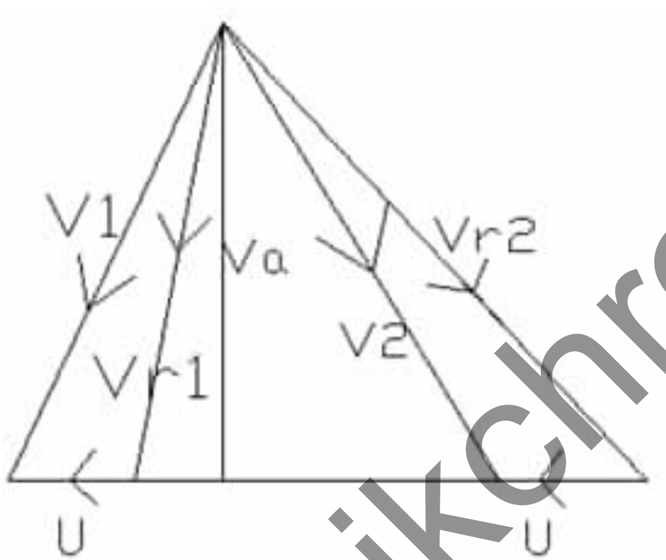
$$\varepsilon_{\text{maximum}} = \frac{\cos^2\alpha_1}{1-R_d\sin^2\alpha_1}$$

For a fixed value of α_1, as Rd increases $\varepsilon_{maximum}$.

But for Rd=1 (100% reaction turbine), this equation doesn't holds good.

Let us examine how $\varepsilon_{maximum}$ is affected by Rd.

Case (1): Rd=1,



$V_{w1}=V_{w2}$

Hence by Euler's turbine equation

$$P = (V_{w1}U_1 - V_{w2}U_2)$$
$$P = U[V_{w1} - V_{w2}]$$

$$P = U_2 V_{w2}$$

$$P = 2UV \cos\alpha_1$$

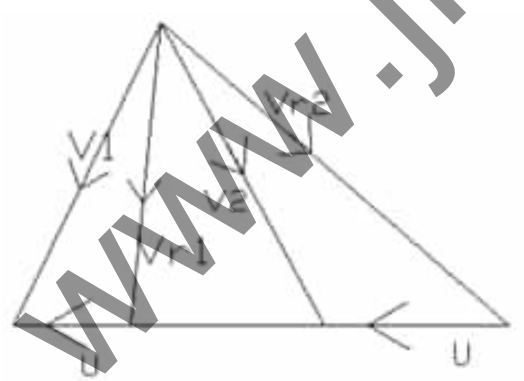
$$\varepsilon = \frac{2UV_1 \cos\alpha_1}{2UV_1 \cos\alpha_1 + \frac{V_1^2}{2}}$$

$$\varepsilon = \frac{2UV_1 \cos\alpha_1}{2UV_1 \cos\alpha_1 \left[1 + \frac{V_2^2}{2(2UV_1 \cos\alpha_1)}\right]}$$

$$\varepsilon = \frac{1}{1 + \frac{V_2^2}{4UV_1 \cos\alpha_1}}$$

For maximum utilization V_2 needs to be axial. If V_2 is to be axial, then V_1 also should be axial which means that the denominator of the expression becomes equal to infinity which reduces to to zero. This only means that α_1 should be as low as possible to get meaning full values of. This represents contradicting condition and hence Rd = 1 is not preferred.

Case (2): Rd>1



V2>V1, Vr2>Vr1

As Rd> ε tends to zero.

In this case, $V_2 > V_1$ and hence V_2 can never be axial and hence the condition for $\varepsilon_{maximum}$ [An axial orientation for V2 can never be met]

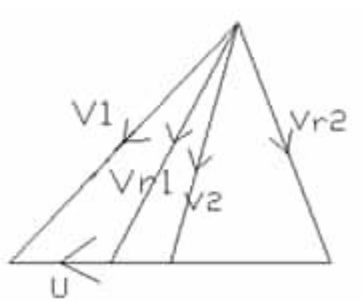
The utilization factor ε is given by

$$\varepsilon = \frac{V_1^2 - V_2^2}{V_1^2 - R_d V_2^2}$$

As Rd increases, ε decreases.

This means that the stator has to function to not only diffuse V_2 to as low a value as possible but also turn the fluid through a very large angle. This results in the poor flow efficiency and hence Rd greater than 100% is not practically preferred.

Case (3): Rd<0 [negative Rd]



$V_{r1} > V_{r2}$

$$R_d = \frac{-(V_{r1}^2 - V_{r2}^2)}{(V_1^2 - V_2^2) - (V_{r1}^2 - V_{r2}^2)}$$

$$\varepsilon = \frac{V_1^2 - V_2^2}{V_1^2 - R_d V_2^2}$$

For this condition, it is noticed that rd is negative denominator increases, ε decreases.

$V_{r2} < V_{r1}$, also means that the pressure is increasing as fluid passes through the rotor.

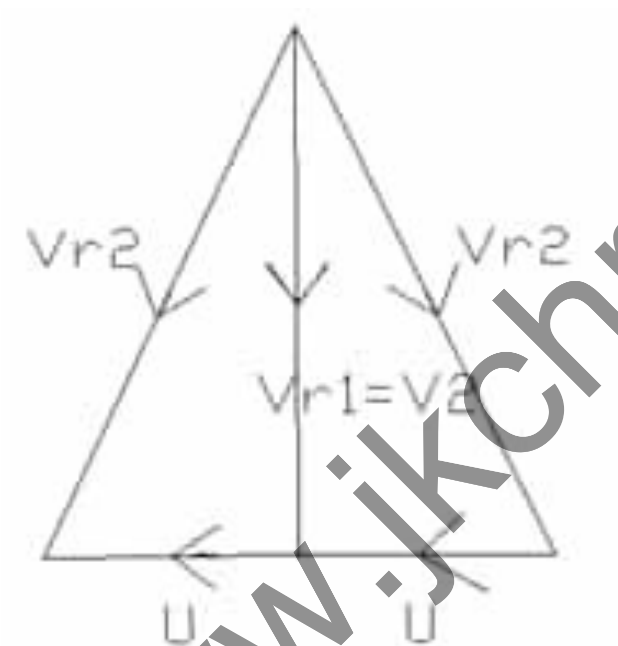
i.e. the rotor is acting like a diffuser. This is not preferred since pressure always has to decrease along the flow path for good flow efficiency. Hence, Rd < 0 is not practically preferred.

Case (4): Rd = 0.5,

We know that for a 50% reaction turbine, velocity triangles are similar and for maximum utilization condition the triangle would appear as shown.

We notice,

$$\alpha_1 = \beta_2$$
$$\alpha_2 = \beta_1$$



The angles are identical but reversed for the rotor and the stator. From the practical view point, the manufacturing of blades becomes simple. Since the same blade can be used for either the stator or the rotor by merely reversing the direction. It can also be shown that in a multistage turbines 50% reaction gives maximum stage efficiency. Since $V_{r2}>V_{r1}$, pressure reduces along the flow path in the rotor resulting in high flow efficiency. In general, n Rd value between 0 and 1 is preferred due to practical considerations.

From the velocity triangle it can be noted that V_{w1}=U

$$P = \frac{m}{t}[V_{w1}U_1 - V_{w2}U_2]$$

$V_{w2} = 0$ (for maximum utilization factor condition)

$\therefore P = U^2$

Comparing the energy transfer achieved by 50% reaction turbine with an impulse turbine when both are designed for $\varepsilon_{maximum}$ condition and operating with the same rotor velocities. We notice that an impulse turbine transfers twice as much energy as 50% reaction turbine gives the better flow efficiencies.

If multi staging is attempted, then for a given value of energy transfer, a 50% reaction turbine would need twice the number of stages as that of impulse turbines. In actual practice, when multistage is attempted, the initial stages are designed for an impulse turbine when maximum fluid velocity is available. The subsequent stages are 50% reaction stages.

Impulse and Reaction Principles

Turbo machines are classified as impulse and reaction machines depending on the relative proportions of the static and dynamic heads involved in the energy transfer. To aid this, we define a term referred to as degree of reaction Rd.

Degree of reaction Rd can be defined as the ratio of static head to the total head in the energy transfer.

$$R_d = \frac{(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)}{(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)}$$

Degree of reaction can be zero, positive or negative.

Rd=0, characterizes a close turbo machine for which a static head is equal to zero.

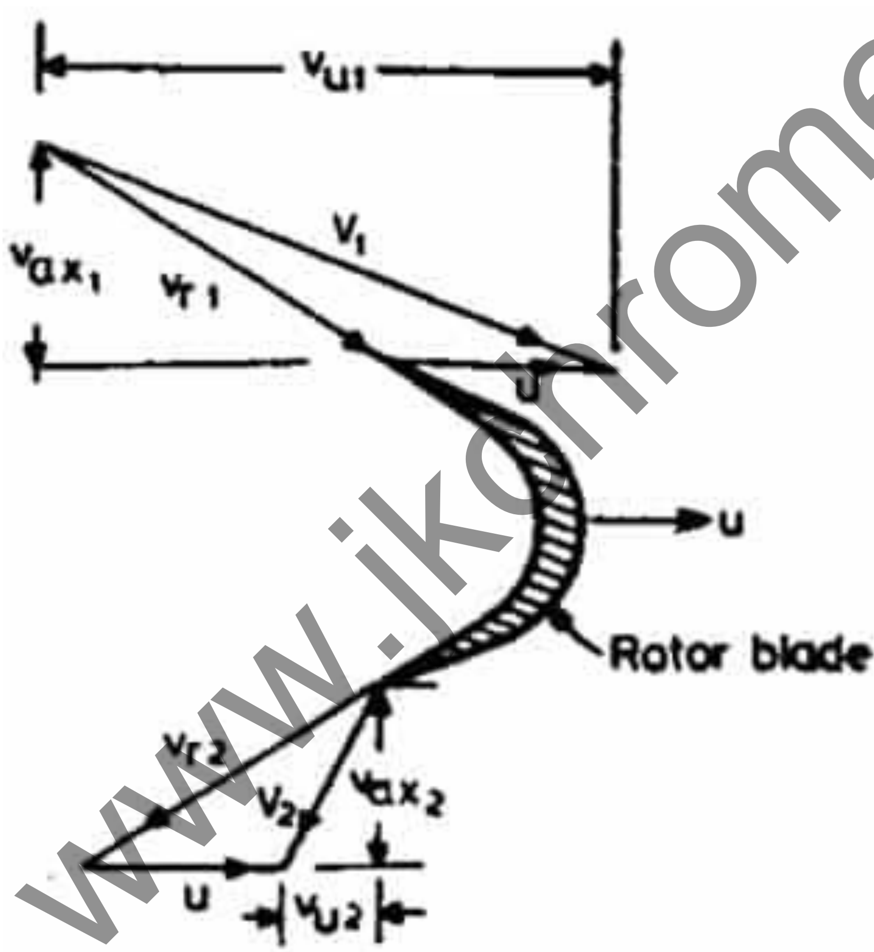
In the most general case, this will happen if $U_1 = U_2$ and $V_{r1} = V_{r2}$.

These classes of turbo machines are referred to as impulse machines. In most practical situations V_{r2} may be less than V_{r1} even though $r_1 = r_2$.

This is generally due to frictional losses. Even then a machine is referred to as an axial flow turbines and pumps would have $r_1 = r_2$ and if $V_{r1} = V_{r2}$, then they become examples of pure impulse machines.

Pelton Wheel, tangential flow hydraulic machines is also example of impulse machine.

Velocity Triangles for impulse machine: Velocity triangle for axial flow impulse machine is shown in the following figure.



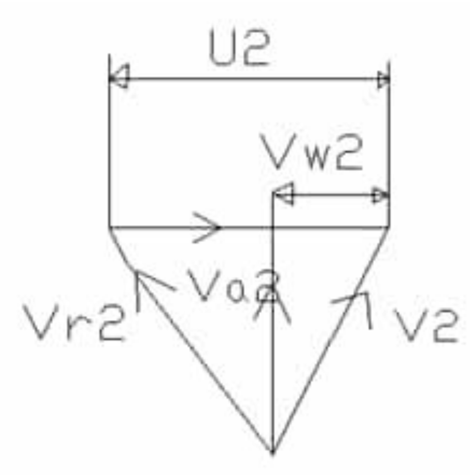
The velocity of whirl at exit is to be calculated by general expression,

$$V_{w2} = U_2 - V_{r2}\cos\beta_2$$

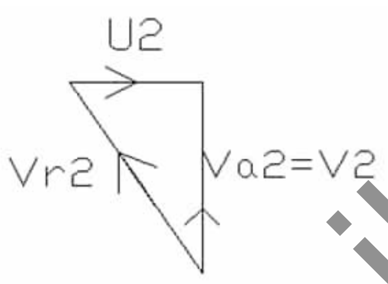
If the value obtained is negative, then it suggests that

$V_{r2} \cos\beta_2 > U_2$

If Vw2 is positive, then OVT would appear as follows:

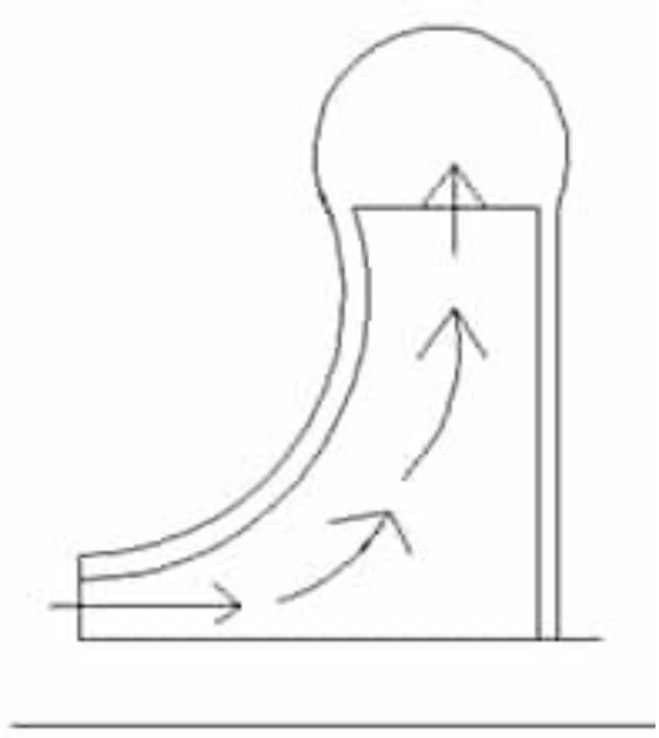


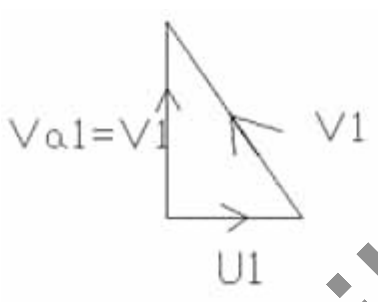
If V2=0, then the OVT would look like

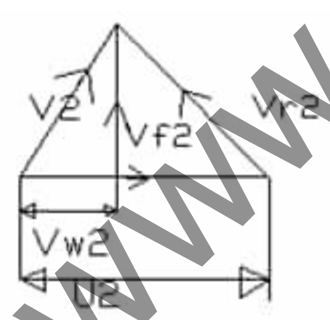


Radial flow Pump and Compressors:

General analysis:




Va1=V1
V1
U1


V2
Vf2
Vr2
Vw2
U2

1.) $V_{w2}U_2 > V_{w1}U_1$

i.e, $U_2 > U_1$ or $r_2 > r_1$ flow is radially outward.

2.) Axial entry $V_1 = V_{a1}$

$\therefore V_{w1} = 0$

3.) $P = \frac{m}{t} V_{w2} U_2$, Watt

$H = \frac{V_{w2}U_2}{g}$ m of fluid.

Most of the turbo machines belong to this class. In general, they have a restricted flow area for a given rotor diameter and have low to medium specific speed.

Significant aspects:

1. Flow is outwards from the smaller to larger radius the Euler's turbine equation. i.e.,
   $P = \frac{m}{t}(V_{w1}U_1 - V_{w2}U_2)$ requires that $V_{w2}U_2 > V_{w1}U_1$ for pumps and compressors which are power absorbing machines. For this sake radial flow compressors and pumps generally have fluid entering at a smaller radius and leaving at a larger radius.
2. The absolute velocity at inlet is oriented parallel to the axes of the shaft i.e., $V_{a1} = V_1$ and hence there is no whirl component at inlet i.e.,$V_{w1} = 0$.
3. Since $V_{w1} = 0$, the energy transferred is purely a function of exit
   $$P = \frac{m}{t} V_{w2} U_2, \text{Watt}$$
   $$H = \frac{V_{w2}U_2}{g} \text{ m of fluid}$$
   condition i.e.

Head-capacity relationship:

$$H = \frac{V_{w2} U_2}{g}$$

$$H = \frac{U_2}{g}\left[U_2 - V_{f2}\cot\beta_2\right]$$

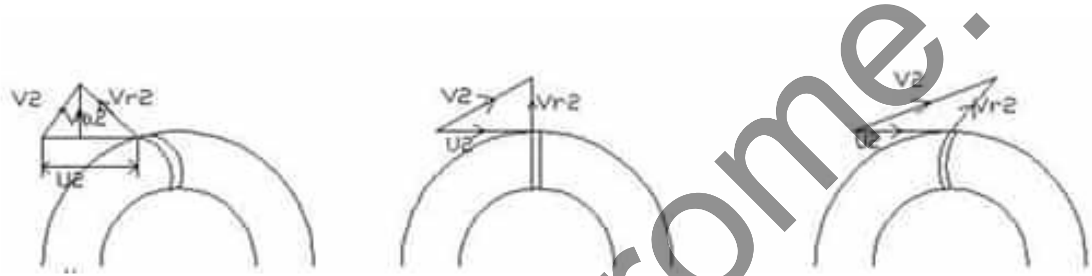
$$Q =_2 = A_2 V_{f2}$$

$$Q = \frac{U_2}{g}\left[U_2 - \frac{Q\cot\beta_2}{A_2}\right]$$

$$H = \left(\frac{U_2^2}{g}\right) - \left(\frac{U_2\cot\beta_2}{g A_2}\right)Q$$

$$H = K_1 - K_2 Q \quad \text{(considering rotor operating for a given speed.)}$$

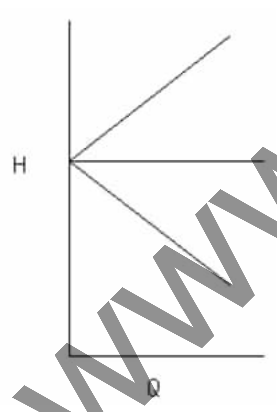
$$K_1 = \frac{U_2^2}{g}$$



$\beta < 90°$ Backward curved vane

$\beta = 90°$ Radial vane

$\beta > 90°$ Forward curved vane



From the velocity triangles for the 3 types of vanes it may be noticed that the whirl component at exit is least for backward curved vane (β<90° and most for a forward curved vane. When operating under similar condition of speed and cross section area. But from a practical view point a high value of exit velocity V2 is not desirable. This is because it becomes necessary to construct a diffuser of unreasonably large dimensions even for moderate sized rotors. Hence backward curved vane with β_2 in the range of 20-25 degrees is preferred for radial flow pumps and compressors. Forward curved vanes are not preferred while radial vanes (β=90°) are used in select applications requiring very high pressure.

Expression for Degree of reaction in terms of rotor velocity and rotor blade angles:

We know that, Degree of reaction is given by,

$$R_d = \frac{(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)}{(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)}$$

$$R_d = \frac{(V_{r2}^2 - V_{r1}^2)}{(V_1^2 - V_2^2) - (V_{r1}^2 - V_{r2}^2)}$$

$$V_{r1}^2 = V_a^2 + (V_a \tan\gamma_1)^2 = V_a^2(1 + \tan^2\gamma_1)$$

$$V_{r2}^2 - V_{r1}^2 = V_a^2(\tan^2\gamma_2 - \tan^2\gamma_1)$$

For a pump it is generally acceptable to write degree of reaction as

$$R_d = \frac{(V_{r1}^2 - V_{r2}^2)}{(V_2^2 - V_1^2) - (V_{r1}^2 - V_{r2}^2)}$$

$$R_d = \frac{V_a^2(\tan^2\gamma_2 - \tan^2\gamma_1)}{(V_1^2 - V_2^2) - (V_{r1}^2 - V_{r2}^2)}$$

We know that, Euler's turbine equation for a pump may be written as

$$U = \frac{1}{g}(V_{w1}u_1 - V_{w2}u_2)$$

$$U = \frac{1}{g}\left[\left(\frac{V_1^2 - V_2^2}{2}\right) - \left(\frac{V_{r1}^2 - V_{r2}^2}{2}\right)\right]$$

Degree of reaction is the ratio of suction head to the total head. Which may be written as

$$R_d = \frac{-\left(\frac{V_{r2}^2 - V_{r1}^2}{2g}\right)}{\frac{1}{2g}\left[\left(V_2^2 - V_1^2\right) - \left(V_{r2}^2 - V_{r1}^2\right)\right]}$$

$$R_d = \frac{\left(\frac{V_{r1}^2 - V_{r2}^2}{2}\right)}{V_{w2}U_2 - V_{w1}U_1}$$

$$R_d = \frac{V_a^2[\tan\gamma_1 + \tan\gamma_2][\tan\gamma_1 - \tan\gamma_2]}{2UV_a[\tan\gamma_1 - \tan\gamma_2]}$$

$$R_d = \frac{V_a^2[\tan\gamma_1 + \tan\gamma_2]}{2UV_a}$$

$$R_d = \frac{V_a}{2U}\frac{[\tan\beta_1 + \tan\beta_2]}{\tan\beta_2 \tan\beta_1}$$

General analysis of Turbines:

They are power generating turbo machines, which run on both incompressible fluids such as water as well as compressible fluids such as gases.

The efficiency of turbines may be defined as the ratio of actual work output to the fluid energy input.

This involves 2 types of efficiencies:

1. **Hydraulic efficiency /isentropic efficiency.**
2. **Mechanical efficiency.**

The mechanical efficiency takes care of all losses due to energy transfer between mechanical elements. In the turbines, mechanical efficiency is very high and of the order of 98 to 99%.

The hydraulic efficiency takes care of losses during flow.

We realize that, turbines must have a residual exit velocity so that flow is maintained.

However, this residual velocity so that flow is it represents a lot far as the rotor is concerned. Hence, even if we have idealized friction free flow it is not possible to transfer all the energy in the fluids due to the need to have the final residual exit velocity.

Hence, hydraulic efficiency is a product of 2 terms and is given by

$\eta_H = \varepsilon * \eta_V$

η_V - where is referred to as vane efficiency and takes care of frictional loss.

Utilization factor:

Utilization factor is defined as the ratio of the actual work transferred from the fluid to the rotor in an ideal condition to the maximum possible work that could be transferred in an ideal condition.

$$\varepsilon = \frac{W_{actual}}{W_{max}} = \frac{(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)}{V_1^2 + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)}$$

$$W_{actual} = (V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)$$

$$W_{actual} = \frac{m}{2}\left[V_1^2 + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)\right]$$

$$W_{max} = W_{actual} + \frac{m}{t}\frac{V_2^2}{2}$$

$$W_{actual} = m[((V_{w1}U_1 - V_{w2}U_2) + \frac{V_2^2}{2}]$$

$$\varepsilon = \frac{(V_{w1}U_1 - V_{w2}U_2)}{(V_{w1}U_1 - V_{w2}U_2) + \frac{V_2^2}{2}}$$

Relationship between ε and R_d:

OR

Derive an expression for ε in terms of Rd:

$$H_s = \frac{1}{2g}[(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]$$

$$H_d = \frac{1}{2g}[V_1^2 - V_2^2]$$

$$R_d = \frac{H_s}{H_s + H_d}$$

$$R_d[H_d + H_s] = H_s$$

$$R_d H_s + R_d H_d = H_s$$

$$H_s = \frac{R_d H_d}{(1 - R_d)}$$

Utilization factor may be written as

$$\varepsilon = \frac{H_s + H_d}{H_s + \frac{V_1^2}{2g}}$$

Which gives,

$$\varepsilon = \frac{V_1^2 - V_2^2}{V_1^2 - R_d V_2^2}$$

This expression holds good for Rd values between 0 and 1. This cannot be used for Rd=1 (100% reaction). Since, the expression becomes equal to 1 suggesting 100% utilization factor which could obviously lead to residual exit velocity V2 becoming zero.

General analysis of Axial flow turbines:

Most turbines involving compressible flow are axial turbines. Generally, steam and gas turbines are axial flow machines.

We know that in all axial turbine machines, $U_1 = U_2 = U$.

And hence the alternative form of turbine equation reduces to

$$P = \frac{m}{t}\frac{1}{2}\left[(V_1^2 - V_2^2) - (V_{r1}^2 - V_{r2}^2)\right]$$

Degree of reaction,

$$R_d = \frac{-(V_{r1}^2 - V_{r2}^2)}{(V_1^2 - V_2^2) - (V_{r1}^2 - V_{r2}^2)}$$

Change of fluid pressure in the rotor happens only due to change in the relative velocity component Vr, since, U remains constant.

Axial flow turbines are of 2 types:

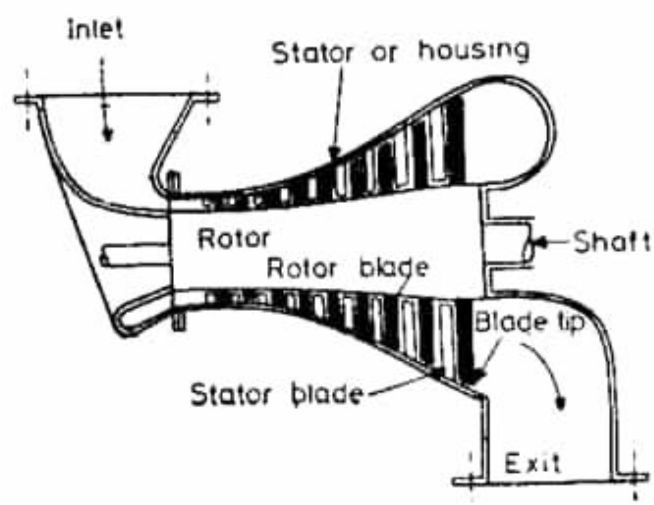
1. Impulse type for which Rd=0.since $V_{r1}=V_{r2}$ and hence power output $P = \frac{m}{t}\frac{1}{2}\left[V_1^2 - V_2^2\right]$
2. Reaction type: Generally any turbine which is not purely an impulse turbine is referred to as a reaction turbine. It is not a 100% reaction turbine. But, it is still referred to as a reaction turbine. Most reaction turbines are designed for 50% reaction which is found to be very advantageous from practical consideration. In the case of steam turbines it is implicit that a reaction turbine is 50% reaction turbine called as parson's reaction turbine.

Turbomachinery

Turbomachine is defined as a device in which energy transfer takes place between a flowing fluid and a rotating element resulting in a change of pressure and momentum of the fluid. Energy is transferred into or out of the turbomachine mechanically by means of input/output shafts.

Principal Parts of a Turbo Machine

1. Rotating element consisting of a rotor on which are mounted blades.
2. A stationary element in the form of guide blades, nozzles, etc.
3. Input/output shafts.
4. Housing



Schematic cross sectional view of a steam turbine showing the principal parts of a turbo machine.

Functions:

1. The rotor functions to absorb/deliver energy to the flowing fluid.
2. The stator is a stationary element which may be of many types:-
    - Guide blades which function to direct the flowing fluid in such a way that energy transfer is maximized.
    - Nozzles which function to convert pressure energy of the fluid to kinetic energy
    - Diffusers which function to convert kinetic energy to pressure energy of the fluid.
3. The input /output shafts function to deliver/receive mechanical energy to or from the machine.
4. The housing is a protective enclosure which also functions to provide a path of flowing fluid. While a rotor & input /output shaft are essential parts of all turbo machines, the stator & the housing are optional.

Classification of Turbo Machines:

1. According to the nature of energy transfer:
    - Power generating turbo machines: In this, energy is transferred from the flowing fluid to the rotor. Hence, enthalpy of the flowing fluid decreases as it flows across. There is a need for an output shaft.

- Ex: Hydraulic turbines such as Francis turbine, Pelton wheel turbine, Kaplan turbine, steam turbine such as De-Laval turbine, Parsons Turbine etc, Gas turbines etc,
- Power absorbing Turbo machines: In this, energy is transferred from the rotor to the flowing fluid. The enthalpy of the fluid increases as it flows there is a need for an input shaft.
- Ex: Centrifugal pump, Compressor, blower, fan etc,
- Power transmitting turbo machines: In this energy is transferred from one rotor to another by means of a flowing fluid. There is a need for an input / output shafts. The transfer of energy occurs due to fluid action.
- Ex: Hydraulic coupling, torque converter etc,

Schematic representation of different types of turbo machine based on fluid flow:

- Axial flow fan.
- Radial outward flow fan.
- Mixed flow hydraulic turbine.

1. **Based on the type of fluid flow:**
    - Tangential flow in which fluid flows tangential to the rotor Ex: Pelton wheel etc,
    - Axial flow in which the fluid flows more or less parallel to the axes of the shafts /rotors. Ex: Kaplan turbine, Axial flow compressor.
    - Radial flow in which fluid flows along the radius of the rotor this is again classified as:
        - Radially inward flow. Ex: Old fancies turbine.
        - Radially outward flow. Ex: Centrifugal Pump
    - Mixed flow which involves radius entry & axial exit or vise-versa. Ex: Modern francises turbine & Centrifugal Pump
2. **Based on the type of Head:**
    - High head &low discharge. Ex: Pelton wheel.
    - Medium head &medium discharge. Ex: Francis turbine.
    - Low head & high discharge. Ex: Kaplan turbine.

Application of 1st & 2nd law of thermodynamics of turbo machines:

In a turbo machine, the fluctuations in the properties when observed over a period of time are found to be negligible. Hence, a turbo machine may be treated as a steady flow machine with reasonable accuracy & hence, we may apply the steady flow energy equation for the analysis of turbo machine.

Hence we may write

$$q - w = (h_2 - h_1) + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)$$

Where, subscript '1' is at the point of entry & subscript '2' is at point of exit.

It is also true that, thermal losses are minimal compared to the amount of work transferred & hence may be neglected. Hence we may write,

$$w = (h_2 + \frac{V_2^2}{2} + gz_2) - (h_1 + \frac{V_1^2}{2} + gz_1)$$

$$-w = (h_{0_2} - h_{0_1})$$

Where, h_{02} & h_{01} are stagnation exit & entry respectively.

$w = \Delta h_{0.}$

In a power generating turbo machine, Δh_0 is negative (since $h_{02} < h_{01}$) & hence w is positive.

On the same line, for a power absorbing turbo machine, Δh_0 is positive (since $h_{02} > h_{01}$) & hence w is negative.

From the 2nd law of Thermodynamics:

$$T\,ds = dh - vdp$$

$$-dw = vdp + T\,ds$$

$$w = -\int_1^2 vdp - \int T\,ds$$

In the above relation, we note that vdp would be a negative quantity for a power generating turbo machine & positive for power absorbing turbo machine.

Hence Tds which is always a positive quantity would reduce the amount of work generated in the former case & increase the work absorbed in the later case.

Efficiency of a turbo machine:

Generally, we define 2 types of turbo machine .in case of turbo machine to account for various losses 2 type of efficiency is considered:

- Hydraulic efficiency/isentropic efficiency
- Mechanical efficiency.

1. **Hydraulic efficiency/isentropic efficiency:**

To account for the energy loss between the fluid & the rotor

$$(\eta_{isen})_{power\ generating\ machine} = \frac{W_{rotor}}{W_{fluid}}$$

$$(\eta_{isen})_{power\ absorbing\ machine} = \frac{W_{fluid}}{W_{rotor}}$$

2. **Mechanical efficiency:**

To account for the energy loss between the rotor & the shaft.

$$(\eta_{mechanical})_{power\ generating\ machine} = \frac{W_{shaft}}{W_{rotor}}$$

$$(\eta_{mechanical})_{power\ absorbing\ machine} = \frac{W_{rotor}}{W_{shaft}}$$

Schematic representation of Compression & Expansion process:

(a) Power absorbing machine. (b) Power generating machine.

(1) Power generating machine

$$W_{act} = h_{01} - h_{02}$$

$$W_{isen} = W_{t-t} = h_{01} - h^1{}_{02}$$

$$W_{t-s} = h_{01} - h^1{}_{02}$$

$$W_{s-t} = h_1 - h^1_{02}$$

$$W_{s-s} = h_1 - h^1_2$$

$$\eta_{t-t} = \frac{W_{act}}{W_{t-t}} = \frac{h_{01} - h_{02}}{h_{01} - h^1_{02}}$$

$$\eta_{t-s} = \frac{W_{act}}{W_{t-s}} = \frac{h_{01} - h_{02}}{h_{01} - h^1_2}$$

$$\eta_{s-t} = \frac{Wact}{Ws - t} = \frac{h_{01} - h_{02}}{h_1 - h^1_{02}}$$

$$\eta_{s-s} = \frac{W_{act}}{W_{s-s}} = \frac{h_{01} - h_{02}}{h_1 - h^1_2}$$

(2). Power absorbing turbomachine :

$$W_{act} = h_{02} - h_{01}$$

$$W_{isen} = W_{t-t} = h^1_{02} - h_{01}$$

$$W_{t-s} = h^1_{02} - h_1$$

$$W_{s-t} = h^1_2 - h_{01}$$

$$W_{s-s} = h^1_2 - h_1$$

$$\eta_{t-t} = \frac{W_{t-t}}{Wact} = \frac{h^1_{02} - h_{01}}{h_{02} - h_{01}}$$

$$\eta_{t-s} = \frac{W_{t-s}}{W_{act}} = \frac{h^1_{02} - h_1}{h_{01} - h^1_2}$$

$$\eta_{s-t} = \frac{Ws - t}{Wact} = \frac{h^1{}_2 - h_{01}}{h_{02} - h_{01}}$$

$$\eta_{s-s} = \frac{W_{act}}{W_{s-s}} = \frac{h^1{}_2 - h_1}{h_{02} - h_{01}}$$

Analysis of Energy Transfer in turbo machines: Analysis of energy transfer in turbo machines requires a consideration of the kinematics and dynamic factors involved. The factors include changes in the fluid velocity, rotor velocity and the forces caused due to change in the velocity.

We apply Newton's second law of motion as applicable to rotary movement. i.e., Torque is proportional to the rate of change of angular momentum.

$$T = \frac{d(mVr)}{dt}$$

Another important consideration is the treatment of a turbo machine as a steady flow machine.

1. This involves following assumptions:
2. Mass-flow rate is constant.
3. State of fluid at any given point does not change.
4. Heat and Work transfer are constant.
5. Leakage losses are negligible.
6. Same steady mass of fluid flows through all section.

Velocity Components:

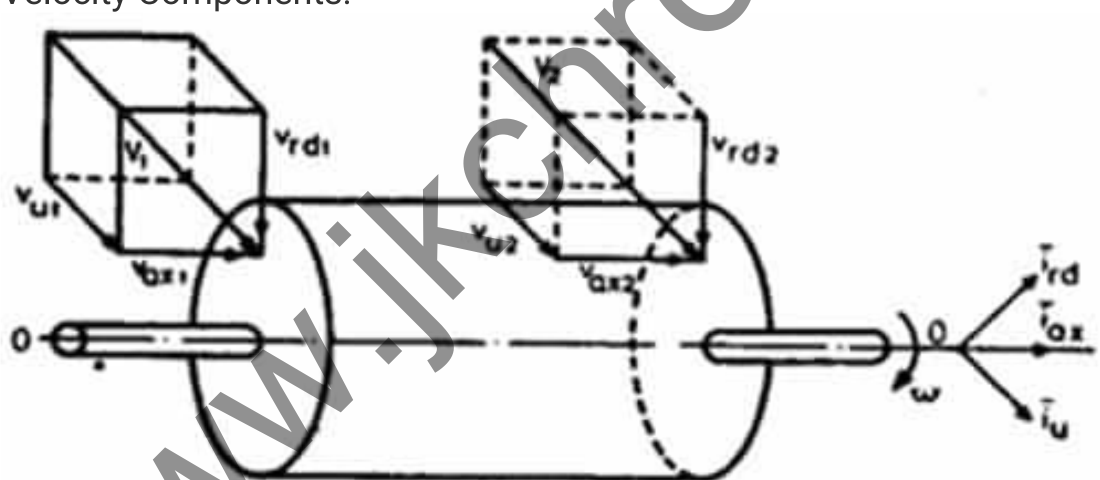
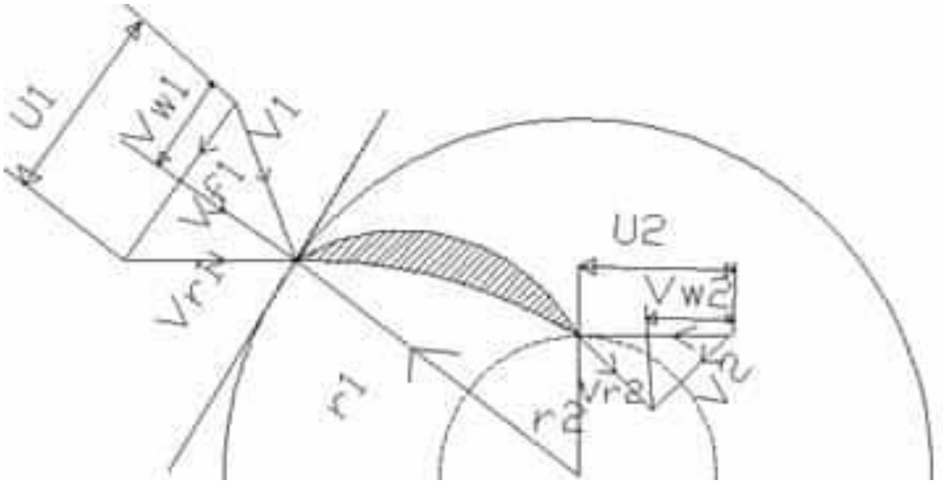


Fig. velocity components through a rotor



The fluid enters the rotor with an absolute velocity say V_1 and leaves with an absolute velocity say V_2.

The absolute velocity of the fluid will have components in the axial, radial and tangential direction which may be referred to as V_a, V_w and V_f respectively.

The axial components do not participate in the energy transfer but cause a thrust which is borne by the thrust bearings. The radial components also do not participate in the energy transfer but cause a thrust which are borne by the journal bearings. The only components which participate in the energy transfer is the tangential component Vw.

V_{a1} and V_{a2} : Axial components of V_1 and V_2 respectively.

V_{f1} and V_{f2} : Radial components of V_1 and V_2 respectively.

V_{w1} and V_{w2} : Tangential components of V_1 and V_2 respectively referred to as whirl velocity, flow velocity. Let the rotor move with an angular velocity ω.

Dimensional Analysis

Dimensional analysis is a mathematical technique which makes use of the study of dimensions as an aid to the solution of several engineering problems. It deals with the dimensions of the physical quantities is measured by comparison, which is made with respect to an arbitrarily fixed value.

Length L, mass M and Time T are three fixed dimensions which are of importance in fluid mechanics. If in any problem of fluid mechanics, heat is involved then the temperature is also taken as fixed dimension. These fixed dimensions are called fundamental dimensions or fundamental quantity.

Secondary or Derived quantities are those quantities which possess more than one fundamental dimension. For example, velocity is defined by distance per unit time (L/T), density by mass per unit volume (M/L^3) and acceleration by distance per second square (L/T^2). Then the velocity, density and acceleration become as secondary or derived quantities. The expressions (L/T), (M/L^3) and (L/T^2) are called the dimensions of velocity, density and acceleration respectively.

Dimensional Analysis

Quantity	Symbol	Dimensions
Mass	m	M
Length	l	L
Time	t	T
Temperature	T	θ
Velocity	u	LT^{-1}
Acceleration	a	LT^{-2}
Momentum/Impulse	mv	MLT^{-1}
Force	F	MLT^{-2}
Energy - Work	W	ML^2T^{-2}
Power	P	ML^2T^{-3}
Moment of Force	M	ML^2T^{-2}
Angular momentum	-	ML^2T^{-1}
Angle	η	$M^0L^0T^0$
Angular Velocity	ω	T^{-1}
Angular acceleration	α	T^{-2}
Area	A	L^2
Volume	V	L^3
First Moment of Area	Ar	L^3
Second Moment of Area	I	L^4
Density	ρ	ML^{-3}
Specific heat-Constant Pressure	C_p	$L^2 T^{-2} \theta^{-1}$

Elastic Modulus	E	$ML^{-1}T^{-2}$
Flexural Rigidity	EI	$ML^{3}T^{-2}$
Shear Modulus	G	$ML^{-1}T^{-2}$
Torsional rigidity	GJ	$ML^{3}T^{-2}$
Stiffness	k	MT^{-2}
Angular stiffness	T/η	$ML^{2}T^{-2}$
Flexibiity	1/k	$M^{-1}T^{2}$
Vorticity	-	T^{-1}
Circulation	-	$L^{2}T^{-1}$
Viscosity	μ	$ML^{-1}T^{-1}$
Kinematic Viscosity	τ	$L^{2}T^{-1}$
Diffusivity	-	$L^{2}T^{-1}$
Friction coefficient	f/μ	$M^{0}L^{0}T^{0}$
Restitution coefficient		$M^{0}L^{0}T^{0}$
Specific heat-Constant volume	C_v	$L^{2}T^{-2}\theta^{-1}$

Dimensional homogeneity

- Dimensional homogeneity means the dimensions of each term in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as the **dimensionally homogeneous** equation.

- The powers of fundamental dimensions i.e., **L, M, T** on both sides of the equation will be identical for a dimensionally homogeneous equation. Such equations are independent of the system of units.
- Let us consider the equation **V = u + at**

  Dimensions of L.H.S = V= L/T = LT-1
  Dimensions of R.H.S = LT^{-1} + (LT^{-2}) (T)
  = LT^{-1} + LT^{-1}
  = LT^{-1}
  Dimensions of L.H.S = Dimensions of R.H.S = LT^{-1}

Therefore, equation **V = u + at** is dimensionally homogeneous

Uses of Dimensional Analysis

- It is used to test the dimensional homogeneity of any derived equation.

- It is used to derive the equation.
- Dimensional analysis helps in planning model tests.

Methods of Dimensional Analysis

- If the number of variables involved in a physical phenomenon is known, then the relationship among the variables can be determined by the following two methods.

Rayleigh's Method

- Rayleigh‟s method of analysis is adopted when a number of parameters or variables is less (3 or 4 or 5).
- If the number of independent variables becomes more than four, then it is very difficult to find the expression for the dependent variable

Buckingham's (Π– theorem) Method

- If there are n – variables in a physical phenomenon and those n-variables contains 'm' dimensions, then the variables can be arranged into (n-m) dimensionless groups called Π terms.
- If f (X_1, X_2, X_3, ……… X_n) = 0 and variables can be expressed using m dimensions then. f (Π1, Π2,Π3, ……… Πn - m) = 0 Where, Π1, Π2, Π3, ……… are dimensionless groups.
- Each Π term contains (m + 1) variables out of which m are of repeating type and one is of non-repeating type.
- Each Π term being dimensionless, the dimensional homogeneity can be used to get each Π term.

Method of Selecting Repeating Variables

- Avoid taking the quantity required as the repeating variable.
- Repeating variables put together should not form a dimensionless group.
- No two repeating variables should have same dimensions.
- Repeating variables can be selected from each of the following properties
  - Geometric property - Length, Height, Width, Area
  - Flow property - Velocity, Acceleration, Discharge
  - Fluid property – Mass Density, Viscosity, Surface Tension

Model Studies

- Before constructing or manufacturing hydraulics structures or hydraulics machines tests are performed on their models to obtain desired information about their performance.
- Models are a small scale replica of actual structure or machine.
- The actual structure is called prototype.

Similitude

- It is defined as the similarity between the prototype and its model. It is also known as similarity. There three types of similarities and they are as follows. •

Geometric similarity

- Geometric similarity is said to exist between the model and prototype if the ratio of corresponding linear dimensions between model and prototype are equal. i.e.

$$\frac{L_p}{L_m} = \frac{h_p}{h_m} = \frac{H_p}{H_m} \ldots\ldots L_r$$

where Lr is known as scale ratio or linear ratio.

$V_m/V_p = \sqrt{(Lr)}$ .... (Velocity Scale Ratio)

$t_m/t_p = \sqrt{(Lr)}$ .....(Time Scale Ratio)

Hence, $a_m/a_p = 1$ .....(Acceleration Scale Ratio)

Kinematic Similarity

- Kinematic similarity exists between prototype and model if quantities such at velocity and acceleration at corresponding points on model and prototype are same.

$$\frac{(V_1)p}{(V_1)m} = \frac{(V_2)p}{(V_2)m} = \frac{(V_3)p}{(V_3)m} \ldots\ldots V_r$$

Where Vr is known as velocity ratio

Dynamic Similarity

- Dynamic similarity is said to exist between model and prototype if the ratio of forces at corresponding points of model and prototype is constant.

$$\frac{(F_1)p}{(F_1)m} = \frac{(F_2)p}{(F_2)m} = \frac{(F_3)p}{(F_3)m} \ldots \ldots \text{F}_r$$

Where Fr is known as force ratio.

Dimensionless Numbers

Following dimensionless numbers are used in fluid mechanics.

- Reynolds's number
- Froude's number
- Euler's number
- Weber's number
- Mach number

Reynolds's number

- It is defined as the ratio of inertia force of the fluid to viscous force.

$$N_{Re} = F_i/F_v$$

Froude's Number (Fr)

- It is defined as the ratio of square root of inertia force to gravity force.

$$F_r = \sqrt{F_i/F_g}$$

Model Laws (Similarity laws)

Reynolds's Model Law

- For the flows where in addition to inertia force, the similarity of flow in the model and predominant force, the similarity of flow in model and prototype can be established if Re is same for both the system.
- This is known as Reynolds's Model Law.

- **R_e for model = R_e for prototype**
- **$(NR_e)m = (NR_e)p$**

$$\left(\frac{\rho VD}{\mu}\right)_m = \left(\frac{\rho VD}{\mu}\right)_p$$

$$\frac{\frac{\rho_m V_m D_m}{\rho_p V_p D_p}}{\frac{\mu_m}{\mu_p}} = 1$$

$$\frac{\rho_r V_r D_r}{\mu_r} = 1$$

Applications

- In the flow of in-compressible fluids in closed pipes.
- The motion of submarine completely under water.
- Motion of airplanes.

Froude's Model Law

- When the force of gravity is predominant in addition to inertia force then similarity can be established by Froude's number.
- This is known as Froude's model law.
- **(Fr)m = (Fr)p**

$$\left(\frac{V}{\sqrt{gL}}\right)_m = \left(\frac{V}{\sqrt{gL}}\right)_p$$

$$\left(\frac{V}{\sqrt{gL}}\right)_r = 1$$

Applications:

- Flow over spillways
- Channels, rivers (free surface flows).
- Waves on the surface.
- Flow of different density fluids one above the other