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Control Systems

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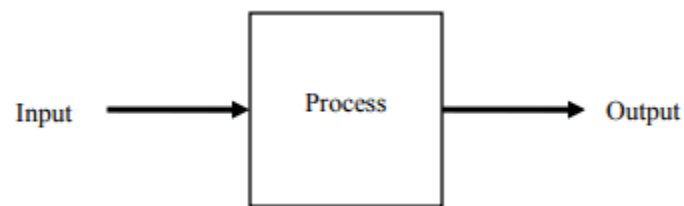
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Basics of Control Systems

Control Engineering

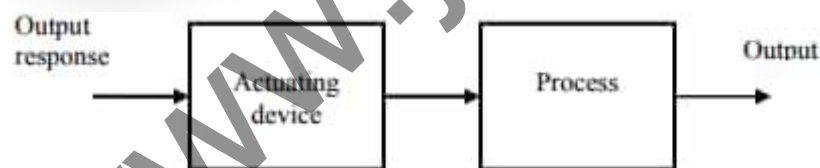
Basically, Control engineering is applicable to aeronautical, chemical, mechanical, environmental, civil, and electrical engineering which is based on the foundations of feedback theory and linear system analysis, and it generates the concepts of network theory and communication system theory.

Hence according to the theory of control engineering, it is not limited to any engineering discipline but applicable to the different areas which require the control process for their functioning & Stable Operations.



Open Loop Control System

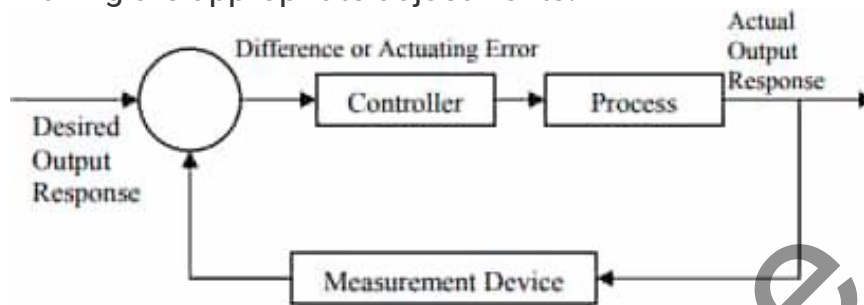
- An open-loop control system consists of a control actuator or controller to receive the desired response.
- It uses a switching device to control the process directly without using any device.
- An illustration of an open-loop control system is an electric toaster.
- In Open Loop Controlling action, there is no feedback system present to sense the error in the desired output.



Closed-Loop Control System

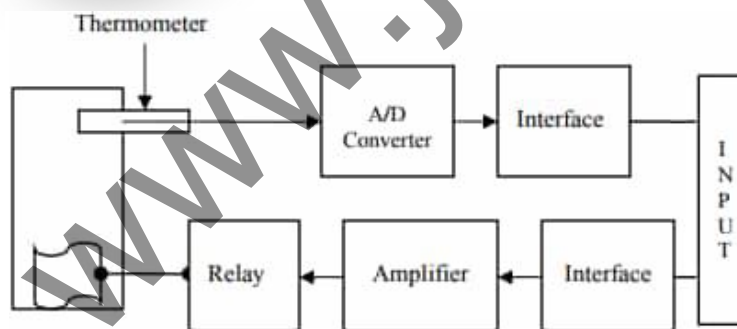
- In a closed-loop control system, it consists of an additional measure of the actual output to compare the actual output with the desired output response.
- This additional measure of the output is called the feedback signal.

- A feedback control system that set out to maintain the relationship of one system variable to another by comparing the functions of these variables and using the difference as a means of control.
- Since the system becomes more complex, the interrelationship of many controlled variables may be considered in the control scheme.
- An example of a closed-loop control system is a person steering (or driving) an automobile by looking at the auto's location on the road and making the appropriate adjustments.



TEMPERATURE CONTROL SYSTEMS

- In the electric furnace, the temperature is measured by a thermometer, which is an analog device.
- The analog temperature converted to digital temperature using an A/D converter. The digital temperature is then fed to a controller through an interface.
- The digital temperature is then compared with the programmed input temperature, and if there is any error, the controller then sends out a signal to the heater, through an amplifier, interface, and relay to bring the furnace temperature to the desired value.



Comparison between Open Loop & Closed Loop Control System

Feature	Open Loop Control System	Closed-Loop Control System
---------	--------------------------	----------------------------

Effect of Output on Input	No effect on input.	The output signal affects the controller output into the system.
Stability	Very Stable	The response changes with the change in the input signal.
Response to external disturbances	No reaction to disturbances. The open Loop control works on fixed output.	The output of the controller adjusts itself in response to the input signal.
Ease of Construction	The controller is easy to construct.	The controller is difficult to construct as it is complex.
Cost	Cheap	Expensive
Bandwidth	Small Bandwidth	Large Bandwidth
Maintenance	Low Maintenance	More Maintenance is required.
Feedback	There is no Feedback	Feedback is always present.

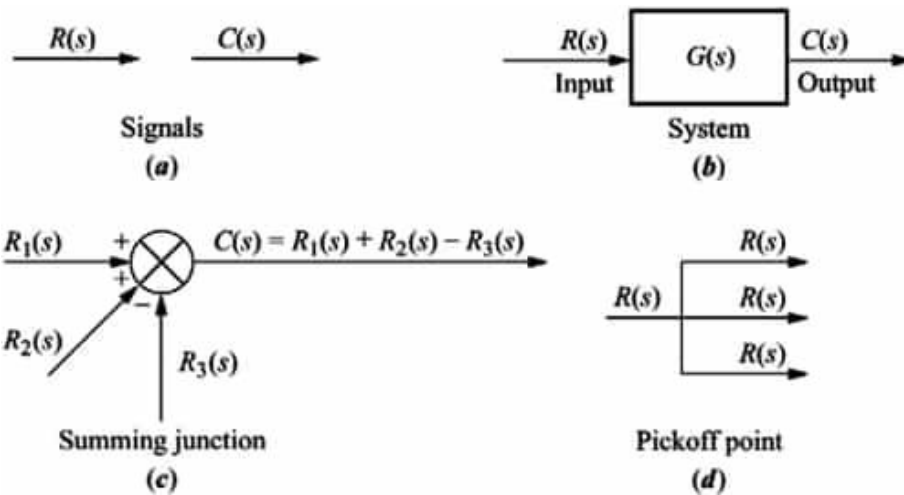
Block Diagram Reduction technique

Need for Block Diagram Reduction:- Some of the block diagrams are complex, such that the evaluation of their performance required simplification (or reduction) of the block diagrams which is done by the block diagram rearrangements.

Advantages of Block Diagram reduction

- Its very simple to construct the block diagram for complicated systems.
- Single, as well as the overall performance of the system, can be studied by using transfer functions shown in the block diagram.
- The overall closed-loop transfer function can be calculated easily using block diagram laws.
- The function of the individual element can be visualized with the help of block diagram.

Components of Linear Time-Invariant Systems

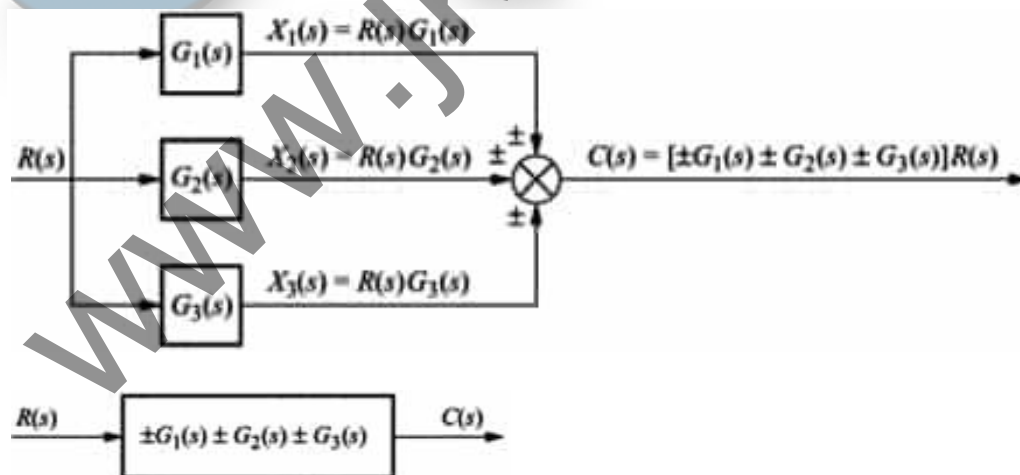


Rules of Block Diagram reduction Technique

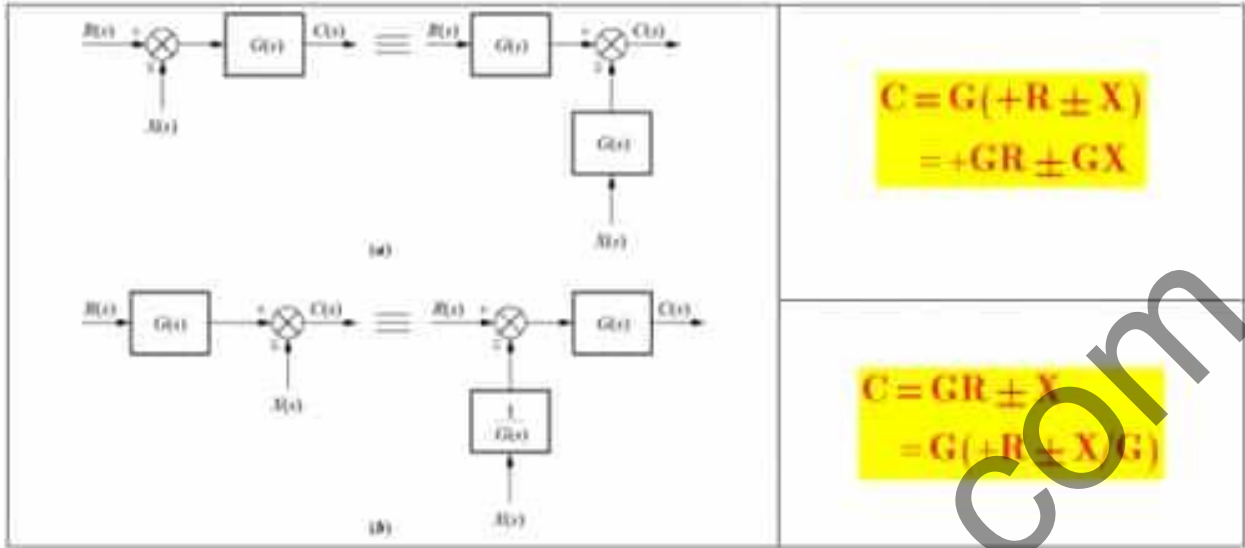
- Cascade series Connection



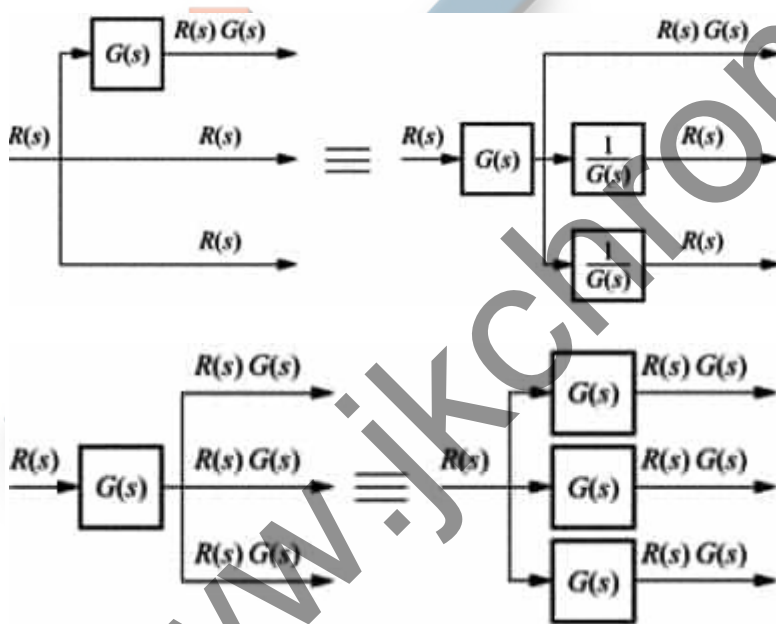
- Parallel Connection



- Block Diagram Algebra for Summing Junctions



• Block Diagram Algebra for Branch Point



Basic Rule For Block Diagram Transformation

	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
1	Combining Blocks in Cascade			$Y = (G_1 G_2) X$
2	Combining Blocks in Parallel; or Eliminating a Forward Loop			$Y = (G_1 \pm G_2) X$
3	Moving a pickoff point behind a block			$y = G u$ $u = \frac{1}{G} y$
4	Moving a pickoff point ahead of a block			$y = G u$
5	Moving a summing point behind a block			$e_2 = G(u_1 - u_2)$
6	Moving a summing point ahead of a block			$y = G u_1 - u_2$ $y = (G_1 - G_2) u$

Meson's Gain Formula

The objectives for this technical note are:

- Define the key terms describing signal-flow graphs.
- Draw signal flow graphs for a block diagram.
- Draw a block diagram from a signal-flow graph.
- List down the steps in the process to solve a system using Mason's gain formula.
- Apply Mason's gain formula to systems in block diagram or signal-flow-graph form.

Basic Definition Related to Meson's Gain Formula

- Forward paths: Forward paths are continuous paths through the graph from the input to the output. No node is passed more than once.

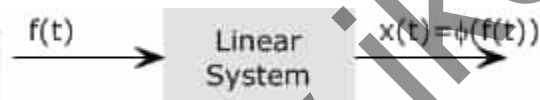
- Feedback loops: Feedback loops are continuous paths through the graph that starts and end at the same node.
- Path gain: Path gain is the product of the signal gains encountered on the path.
- Loop gain: Loop gain is the product of signal gains encountered in a feedback loop.
- Source node: Source nodes are nodes with only outgoing branches.
- Sink nodes: Sink nodes are nodes with only incoming branches.

Mesons gain Formula Statement

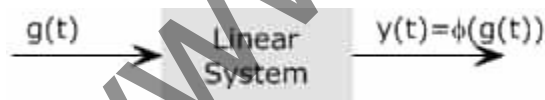
Mason's Gain Rule states $T = \frac{\sum_k P_k \Delta_k}{\Delta}$ where,

- P_k represents the path gain for the kth forward path.
- $\Delta = 1 - (\sum \text{Sum of all individual loop gains}) + (\sum \text{sum of products of all pairs of loop gains, (non-touching loops)}) - (\sum \text{sum of products of all triples of loop gains, (Non-touching loops)}) + \dots$
- $\Delta_k = \Delta - (\sum \text{loop gains in } \Delta \text{ that touch forward path } k)$
- **Linear System:** Essentially a Linear system is one that follows the principle of **Superposition & Homogeneity** in their response to the system.

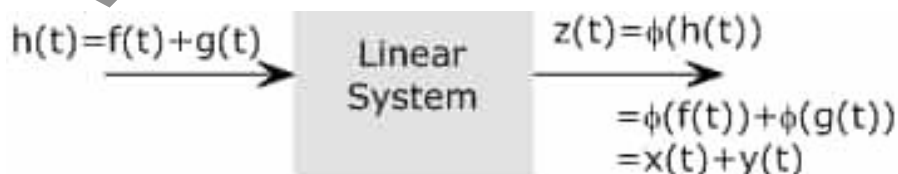
Consider a system with the input $f(t)$ and output $x(t)$



Now if the input is changed to $g(t)$, the output is $y(t)$



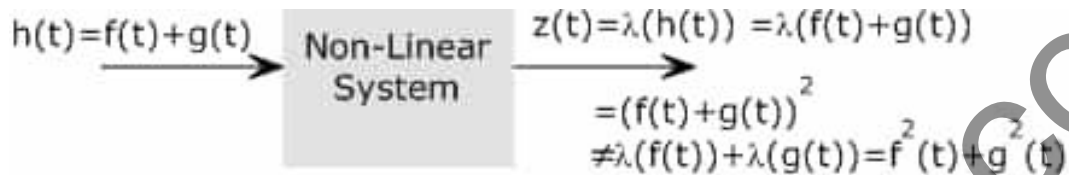
If the system is linear, then an input of $h(t) = g(t) + f(t)$ yields an output $z(t) = x(t) + y(t)$



- **Example of a nonlinear system:** Now consider the same situation when the system is nonlinear for example a squaring function.



Now we can not find the output to the complex function $h(t)$ by adding the responses of the systems to the simpler function.



- **Time-Invariant:** In the **time-invariant system**, the physical parameters of the system do not change with time. The classic example of a non-time invariant (or time-variant) system is a rocket whose mass changes with time (a time-invariant rocket would have constant mass).
- **Continuous-Time:** The Continuous-time systems are time is a continuous, or real-valued, variable. On the other hand, *discrete-time* systems have time that moves in discrete steps.

Examples of discrete-time systems include weekly closing stock prices (updated weekly), the sound on a standard audio CD (updated 44,100 times per second).

Analogous System

An analogous electrical and mechanical system has differential equations of the same kind. There are two analogies that are used to go between the electrical and mechanical systems.

Electrical Quantity	Mechanical Analog I (Force-Current)	Mechanical Analog II (Force Voltage)
Voltage, e	Velocity, v	Force, f
Current, i	Force, f	Velocity, v
Resistance, R	Lubricity, $1/B$ (Inverse friction)	Friction, B
Capacitance, C	Mass, M	Compliance, $1/K$ (Inverse spring constant)
Inductance, L	Compliance, $1/K$ (Inverse spring constant)	Mass, M
Transformer, $N1:N2$	Lever, $L1:L2$	Lever, $L1:L2$

To understand the analogy more clearly. The parameters for the mechanical analogous are formed by substituting the analogous parameters into the equations for the electrical elements. For example, by Ohm's law for electrical circuit $e=iR$. For the Mechanical analog, e is replaced by v , i by f and R by $1/B$, which yields $v=f/B$.

Electrical Equation	Mechanical Analog I (Force-Current)	Mechanical Analog II (Force Voltage)
$e = iR$	$v = \frac{f}{B}$	$f = vB$
$e = L \frac{di}{dt}$ $i = \frac{1}{L} \int e \cdot dt$	$v = \frac{1}{K} \frac{df}{dt}$ $f = K \int v \cdot dt = K \cdot x$	$f = M \frac{dv}{dt} = M \cdot a$ $v = \frac{1}{M} \int f \cdot dt$
$e = \frac{1}{C} \int i \cdot dt$ $i = C \frac{de}{dt}$	$v = \frac{1}{M} \int f \cdot dt$ $f = M \frac{dv}{dt} = M \cdot a$	$f = K \int v \cdot dt = K \cdot x$ $v = \frac{1}{K} \frac{df}{dt}$
power = $e \cdot i$	power = $v \cdot f$	power = $f \cdot v$
Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2} = \frac{i_2}{i_1}$	Lever $\frac{v_1}{v_2} = \frac{L_1}{L_2} = \frac{f_2}{f_1}$	Lever $\frac{f_1}{f_2} = \frac{L_2}{L_1} = \frac{v_2}{v_1}$
capacitor energy $\frac{1}{2} C \cdot e^2$	mass energy $\frac{1}{2} M \cdot v^2$	spring energy $\frac{1}{2} K \cdot x^2 = \frac{1}{2} K \cdot \left(\frac{f}{K}\right)^2 = \frac{1}{2} \frac{f^2}{K}$
inductor energy $\frac{1}{2} L \cdot i^2$	spring energy $\frac{1}{2} K \cdot x^2 = \frac{1}{2} K \cdot \left(\frac{f}{K}\right)^2 = \frac{1}{2} \frac{f^2}{K}$	mass energy $\frac{1}{2} M \cdot v^2$
$\sum_{node} \text{current} = 0$	$\sum_{object} \text{forces} = 0$	$\sum_{loop} \text{velocities} = 0$ (not explicitly used)
$\sum_{loop} \text{voltage} = 0$	$\sum_{loop} \text{velocities} = 0$ (not explicitly used)	$\sum_{object} \text{forces} = 0$

Force-Current 'F-I' Analogy (Electrical to Mechanical)

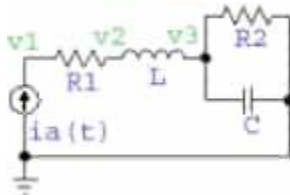
Kirchoff's Current Law and D'Alemberts Law (with inertial forces included) are helpful for converting an electrical circuit to a mechanical system.

Electrical	Mechanical 1
capacitor energy $\frac{1}{2}C \cdot e^2$	mass energy $\frac{1}{2}M \cdot v^2$
voltage of ground=0 <i>(you can apply any current to ground and voltage remains 0)</i>	velocity of ground=0 <i>(you can apply any force to ground and velocity remains 0)</i>

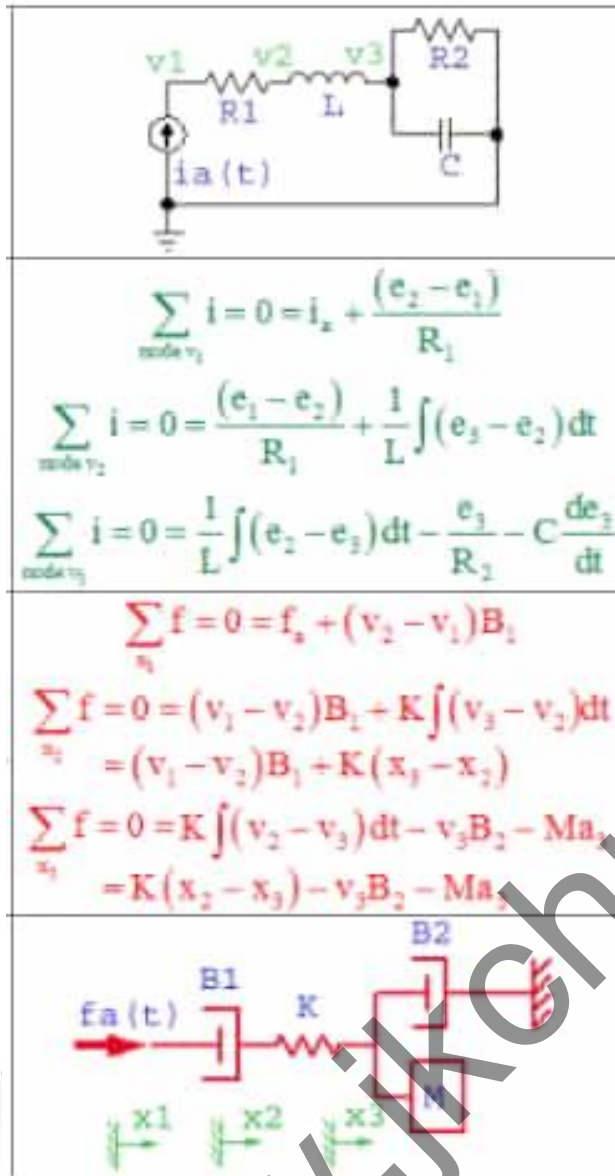
Procedure for Conversion from Electrical to Mechanical.

- Start with an electrical circuit. Label all node voltages.
- Write a node equation for each node voltage.
- Rewrite the equations using analogous making substitutions from the table, with each electrical node being replaced by a position.
- Draw the mechanical system that interconnects with the equations.

Example: Draw the mechanical equivalent circuit of the given system.

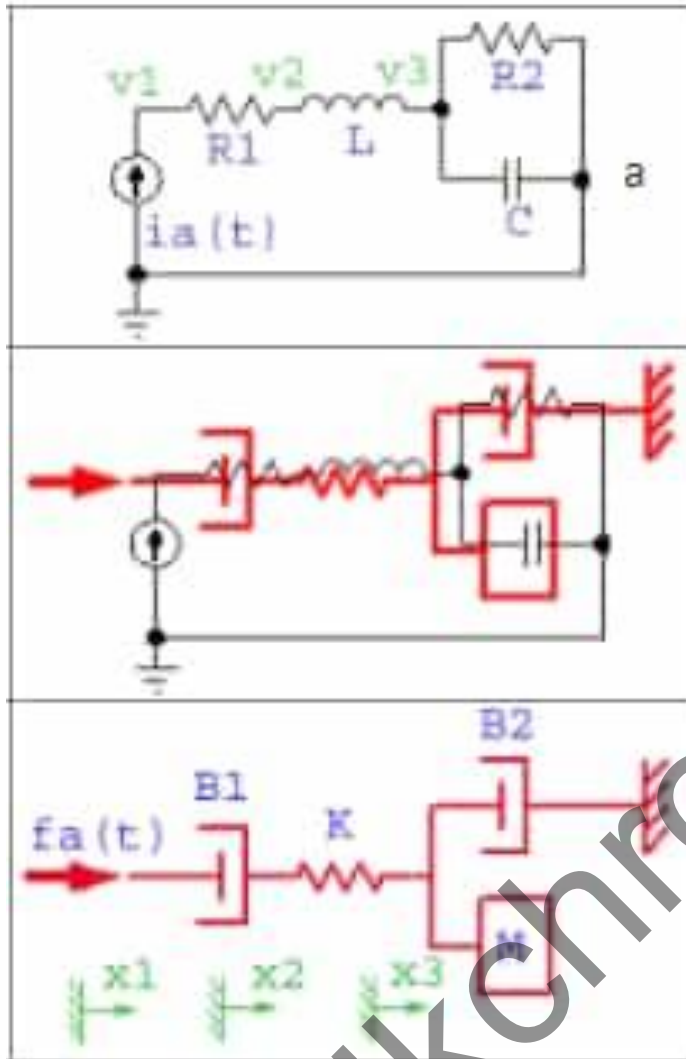


Solution: By following the steps in the given Procedure



Alternative method: Another way from electrical to mechanical simply redrawing the electrical circuit using mechanical components.

- Draw over the circuit, replacing electrical elements with their analogous; voltage sources by input velocities, current sources replaced by force generators, resistors with friction elements, inductors with springs, and capacitors (which must be grounded) by masses. Each node becomes a position (or velocity)
- Label positions, currents, and the mechanical elements as they were in original electrical circuits.



Force-Current 'F-I' to Electrical to Mechanical

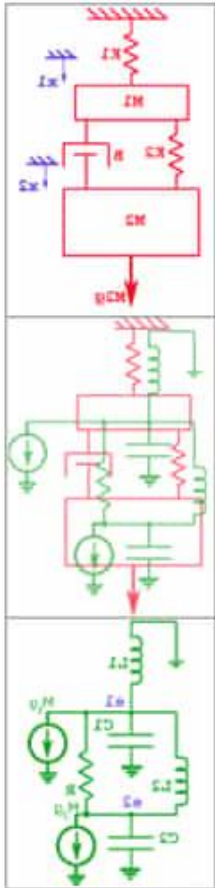
The procedure for Mechanical to Electrical analogy is simply the reverse of Electrical to Mechanical analogy. Either a mathematical method can be used as in the previous example, Electrical to Mechanical conversion can be understood by reading the table from bottom to top, or by the method where force generators are replaced by current sources, friction elements by resistors, springs by inductors, and masses by capacitors (which are grounded). Each position becomes a node in the circuit.

Procedure for 'F-I' analogy for Electrical to Mechanical Conversion

- Start with the mechanical system. Label all positions.
- Draw the circuit by replacing mechanical elements with their analogous; force generators by current sources, input velocities by voltage

sources, friction elements by resistors, springs by inductors, and masses by capacitors (which are grounded). Each position becomes a node.

- Label the nodes and electrical elements as they were in the original mechanical system.



Force-Voltage 'F-V' Analogy (From Electrical to Mechanical)

The important relationship when converting from a circuit to the Mechanical analog is that between Kirchoff's Voltage Law and D'Alemberts Law (with inertial forces included).

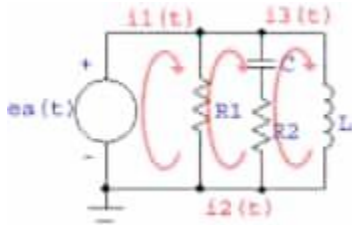
Electrical	Mechanical 2
$\sum \text{Voltage} = 0$	$\sum \text{forces} = 0$

Procedure for Conversion from Electrical to Mechanical

- Start with an electrical circuit. Label all currents. Choose the currents so that only one current flows through the inductors.

- Write loop equations for each loop.
- Rewrite the equations using analogs, making substitutions from the table, with each electrical loop being replaced by a position.
- Draw the mechanical system that interconnects with the equations.

Example: Draw the Mechanical equivalent system of the Electrical Circuit.



Solution: By following the procedure given for force voltage analogy.



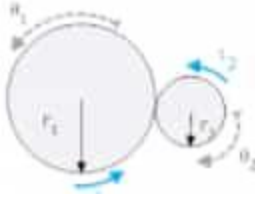
$\sum_{\text{all}} v = 0 = e_s + (i_2 - i_1)R_2$ $\sum_{\text{all}} v = 0 = (i_1 - i_2)R_1 + \frac{1}{C} \int (i_1 - i_2) dt + (i_1 - i_2)R_2$ $\sum_{\text{all}} v = 0 = \frac{1}{C} \int (i_2 - i_1) dt + (i_2 - i_1)R_2 - L \frac{di_2}{dt}$
$\sum_{\text{all}} f = 0 = f_s + (v_2 - v_1)B_1$ $\sum_{\text{all}} f = 0 = (v_1 - v_2)B_1 + K \int (v_1 - v_2) dt + (v_1 - v_2)B_2$ $= (v_1 - v_2)B_1 + K(x_1 - x_2) + (v_1 - v_2)B_2$ $\sum_{\text{all}} f = 0 = K \int (v_2 - v_1) dt + (v_2 - v_1)B_2 + M \frac{dv_1}{dt}$ $= K(x_2 - x_1) + (v_2 - v_1)B_2 + M a_1$

Rotating Mechanical Systems

Gear System

The gear system performs many functions, here we look at the gears that increase or decrease angular velocity (while simultaneously decreasing or increasing torque, such that energy is conserved).

- If we consider two gears in equilibrium and in contact with each other,
- we can obtain very useful relationships.

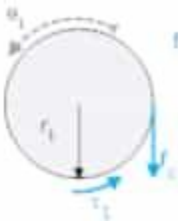


- First, we note the geometric relationship that concludes from the path that the arc lengths along their circumference must be equal to the gear's turn.



Since the arc lengths (shown with a heavy blue line) must be equal \Rightarrow arc length $\Rightarrow r_1\theta_1 = r_2\theta_2$ arc length

- Now we can derive the second relationship from a torque balance. Here we must define a force between the gears termed a "contact force." This force must be equal and opposite across the interface between the two gears, but its direction is arbitrary.



Since the contact force is tangent to both gears and so produces a torque equal to the radius times of the force.

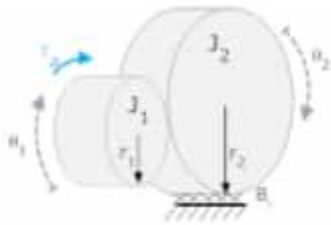


We can have a torque balance on each of the two gears

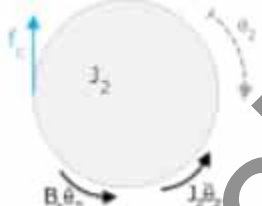
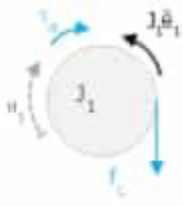
For Gear 1: torque $T_1 = f_c r_1$ or $f_c = T_1 / r_1$ & **For Gear 2:** torque $T_2 = f_c r_2$ or $f_c = T_2 / r_2$

From the above two equations we concluded that $f_c = T_1/r_1 = T_2/r_2$

- In the system below, a torque (T_a), is applied to gear 1 (with a moment of inertia J_1). It, in turn, is connected to gear 2 (with a moment of inertia J_2) and a rotational friction B_r . The angle θ_1 is defined as positive clockwise, θ_2 is also defined as positive clockwise. The torque acts in the direction of θ_1 .



- We start by drawing free body diagrams, including a contact force that we will arbitrarily choose to be down on J_1 and up on J_2 . The directions of the reactive forces due to inertia and friction are chosen, and as always, opposite to the defined positive direction.



This yields the two equations of motion

$$T_a + f_c r_1 - J_1 \theta_1 = 0$$

$$f_c r_2 - J_2 \theta_2 + B_r \theta_2 = 0$$

- We can easily solve for f_c and eliminate it from the above equations, but we also need to eliminate θ_2 . To do this, we use the relationship between θ_1 and θ_2 (from equal arc lengths).

$$r_1 \theta_1 = -r_2 \theta_2$$

- Note that we have a negative sign here because of the way θ_1 and θ_2 were defined (if θ_1 moves in the positive direction, then θ_2 is negative). When you use the arc length expression, you must be careful of the signs.

$$f_c r_2 - J_2 \theta_2 - B_r \theta_2 = 0$$

$$\text{and } \theta_2 = -r_1/r_2(\theta_1)$$

we can we write it as $f_c r_2 - J_2 \{r_1/r_2(\theta_1)\} = 0$

$$f_c = -J_2 \{r_1/(r_2)^2\} \theta_1 - B r \{r_1/(r_2)^2\} \theta_1$$

We can put this into the equation for J1 and solve (in standard form with the output (θ_1) on the left, and the input (T_a) on the right).

$$T_a + f_c r_1 - J_1 \theta_1 = 0$$

$$\text{or } T_a + [-J_2 \{r_1/(r_2)^2\} \theta_1 - B r \{r_1/(r_2)^2\} \theta_1 r_1] - J_1 \theta_1 = 0$$

$$\text{or } \{J_1 + J_2 (r_1/r_2)^2\} \theta_1 + B r (r_1/r_2)^2 \theta_1 = T_a$$

$$\text{So we get } \theta_1 = r_2/r_1(\theta_2) \text{ \& } \omega_1 = r_2/r_1(\omega_2)$$

$$\text{\& Since from the relation } T_1 = r_1/r_2(T_2)$$

we concluded that

$$T_1 \omega_1 = \{r_1/r_2(\tau_2)\} \{r_2/r_1(\omega_2)\} = T_2 \omega_2$$

$$T_1 \omega_1 = T_2 \omega_2$$

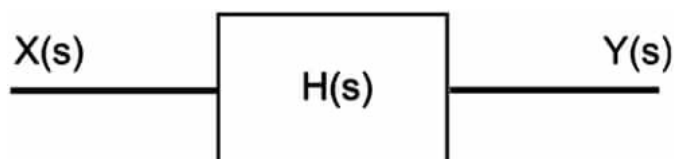
Transfer Function

Transfer Function

The Transfer Function (TF) of a System is the ratio of the output to the input of a system, in the Laplace domain considering its initial conditions and equilibrium point to be zero.

If we have an input function of $X(s)$, and an output function $Y(s)$, we define the transfer function $H(s)$.

$$H(s) = \frac{Y(s)}{X(s)}$$



Impulse Response

Since we know that in the time domain, generally, we define the input to a system as $x(t)$ and the output of the system as $y(t)$. The relationship between the input and the output is represented as the impulse response, $h(t)$.

We can use the following equation to define the impulse response:

$$h(t) = \frac{y(t)}{x(t)}$$

- The Impulse Function, denoted with $\delta(t)$ is a function defined piece-wise as follows

$$\delta(t) = \begin{cases} 0, & t < 0 \\ \text{undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

- An examination of the impulse function shows that it is related to the unit-step function as follows

$$u(t) = \int \delta(t) \quad \text{or} \quad \delta(t) = \frac{du(t)}{dt}$$

- The impulse response (IR) must always satisfy the following condition, or else it is not a true impulse function

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Convolution in Time Domain

If we have the system input and the impulse response of the system, we can calculate the system output using the convolution operation as such as

$$y(t) = h(t) * x(t).$$

Time-Invariant System Response

If the system is time-invariant, then the general description of the system can be replaced by a convolution integral of the system's impulse response and the system input. We can call this the convolution description of a system.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution Theorem

- Convolution in the time domain turns into multiplication in the complex Laplace domain.
- Multiplication in the time domain turns into convolution in the complex Laplace domain.

$$L[f(t)*g(t)] = F(s)G(s)$$

$$L[f(t)g(t)] = F(s)*G(s)$$

Result: If the complex Laplace variable is s , then we generally denote the transfer function of a system as either $G(s)$ or $H(s)$. If the system input is $X(s)$, and the system output is $Y(s)$, then the transfer function can be defined as such

$$H(s) = \frac{Y(s)}{X(s)}$$

- So if we know the input to a given system, and we have the transfer function of the system, we can solve for the system output by multiplying as

$$Y(s) = H(s)X(s)$$

Example: Impulse Response

- Since we know that the Laplace transform of the impulse function, $\delta(t)$ is $L[\delta(t)] = 1$
- So, when we put this into the relationship between the input, output, and transfer function, we get

$$Y(s) = X(s)H(s)$$

$$\text{or } Y(s) = (1)H(s)$$

$$\text{or } Y(s) = H(s)$$

In other words, the "impulse response" is the o/p of the system when we input an impulse function.

Transfer Functions for Linear Systems

Consider a linear input/output system described by the controlled differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \dots + b_m u$$

where u is the input and y is the output.

- To obtain the transfer function of the system of the last equation let us input be $\mathbf{u(t) = e^{st}}$.
- Since the system is linear, there is an output of the system that is also an exponential function $\Rightarrow \mathbf{y(t) = Y_0 e^{st}}$.
- Inserting the signals into the last equation, we find

$$(s^n + a_1 s^{n-1} + \dots + a_n) y_0 e^{st} = (b_0 s^m + b_1 s^{m-1} + \dots + b_m) e^{-st}$$

- And the response of the system can be completely described as

$$\mathbf{a(s) = s^n + a_1 s^{n-1} + \dots + a_n \ \& \ b(s) = b_0 s^m + b_1 s^{m-1} + \dots + b_m}$$

$$y(t) = y_0 e^{st} = \frac{b(s)}{a(s)} e^{st}$$

- So the transfer function of the given function is given by

$$G(s) = \frac{b(s)}{a(s)}$$

Transfer Function Alternative Method

To investigate how a linear system responds to the exponential input $u(t) = e^{st}$ we consider the state space system

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Let the input signal be $u(t) = e^{st}$ and assume that $\mathbf{s \neq \lambda_i(A)}$, where $i = 1, \dots, n$, where $\lambda_i(A)$ is the i th eigenvalue of A . then

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\mathbf{B}e^{s\tau}d\tau = e^{At}\mathbf{x}(0) + e^{At}\int_0^t e^{(sI-A)\tau}\mathbf{B}d\tau$$

Since $s \neq \lambda(A)$ the integral can be evaluated and we get

$$\begin{aligned}\mathbf{x}(t) &= e^{At}\mathbf{x}(0) + e^{At}(sI - A)^{-1}\left[e^{(sI-A)\tau}\right]_{\tau=0}^t \mathbf{B} \\ &= e^{At}\mathbf{x}(0) + e^{At}(sI - A)^{-1}\left(e^{(sI-A)t} - I\right)\mathbf{B} \\ &= e^{At}\left(\mathbf{x}(0) - (sI - A)^{-1}\mathbf{B}\right) + (sI - A)^{-1}\mathbf{B}e^{st}\end{aligned}$$

& Output $y(t) = C\mathbf{x}(t) + Du(t)$

$$= Ce^{At}\{\mathbf{x}(0) - (sI - A)^{-1}\mathbf{B}\} + [D + C(sI - A)^{-1}\mathbf{B}]e^{st}$$

Note: One term of the output is proportional to the input $u(t) = e^{st}$. This term is called the pure exponential response.

- If the initial state is chosen as $\mathbf{x}(0) = (sI - A)^{-1}\mathbf{B}$
- the output only consists of the only exponential response and both the state and the output are directly proportional to the input

$$\mathbf{x}(t) = (sI - A)^{-1}\mathbf{B}e^{st} = (sI - A)^{-1}\mathbf{B}u(t)$$

$$y(t) = [C(sI - A)^{-1}\mathbf{B} + D]e^{st} = [C(sI - A)^{-1}\mathbf{B} + D]u(t).$$

- The ratio of the output and the input

$G(s) = C(sI - A)^{-1}\mathbf{B} + D$ is the transfer function of the system.

- For Homogeneous Equation i.e; $D=0$ then

$G(s) = C(sI - A)^{-1}\mathbf{B}$ is the transfer function of the system.

The Concept of Pole & Zero

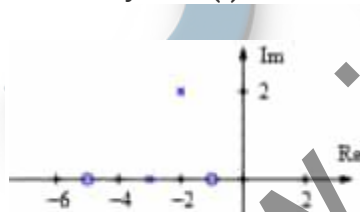
- Poles and Zeros of a transfer function are the frequencies for which the value of the transfer function becomes infinity or zero respectively.
- The transfer function has many useful explanations and the features of a transfer function are often associated with important system properties. Three of the most important features are the gain and the locations of the poles and zeros.

$$H(s) = \frac{N(s)}{D(s)}$$

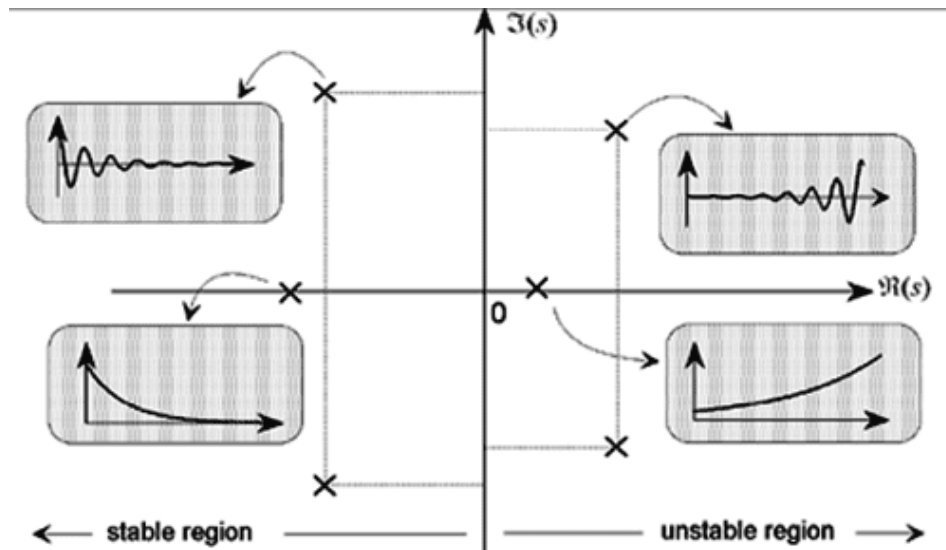
- In the above equation, $N(s)$ and $D(s)$ are simple polynomials in s . **Zeros** are the **roots of $N(s)$** by setting $N(s)=0$ and solving for s . **Poles** are the **roots of $D(s)$** , by setting $D(s)=0$ and solving for s .
- In general **Transfer Function** must not have more zeros than poles, we can state that the polynomial order of $D(s)$ must be greater than or equal to the polynomial order of $N(s)$.

Effects of Poles and Zeros

- when s approaches a **zero (Zeros of Transfer function)**, the **numerator** of the transfer function $N(s) \rightarrow 0$.
- When s approaches a **pole (Poles of Transfer Function)** the **denominator** of the transfer function $D(s) \rightarrow 0$ & the value of the transfer function approaches infinity.
- An output value of infinity should raise an alarm bell for people who are familiar with BIBO stability.
- Poles & Zeros of a transfer function are represented in **S-Plane**, In s -plane $s = \sigma \pm j\omega$ where σ represent the attenuation on X-axis & ω represents the angular velocity represented on the y-axis.
- In the s -plane, the Poles are located by a cross (*) & the Zeros are located by dot (.)



- For a Stable System, All the Poles & Zeros of a Transfer Function must be lie in the left half of the s -plane.



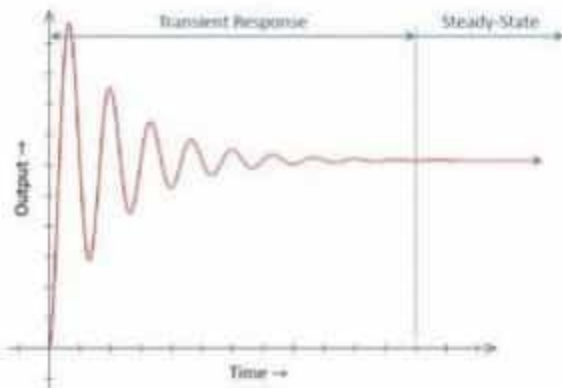
Note: If the **Poles or Zero** lie on the **imaginary** axis then it must be simple (**order only 1**) then the system is said to be **Marginally Stable**, If the Order of multiplicity of Poles or Zeros on Imaginary Axis is more than 1 in that case system will become Unstable.

Time Domain Analysis

The Time Domain Analyzes of the system is to be done on basis of time. The analysis is only be applied when the nature of the input plus the mathematical model of the control system is known. Expressing the main input signals is not an easy task and cannot be determined by simple equations. There are two components of any system's time response, which are: Transient response & Steady-state response.

- **Transient Response:** This response is dependent upon the system poles only and not on the type of input & it is sufficient to analyze the transient response using a step input.
- **Steady-State Response:** This response depends on system dynamics and the input quantity. It is then examined using different test signals by the final value theorem.

$$c(t) = ctr(t) + css(t)$$



Standard Test Input Signals

Step signal: $r(t) = Au(t)$.

Ramp signal: $r(t) = At; t > 0$.

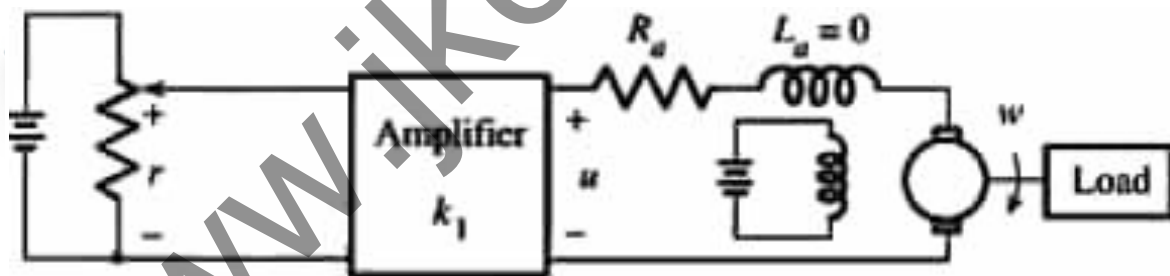
Parabolic signal: $r(t) = At^2/2; t > 0$.

Impulse signal: $r(t) = \delta(t)$.



Time-Response of First-Order System

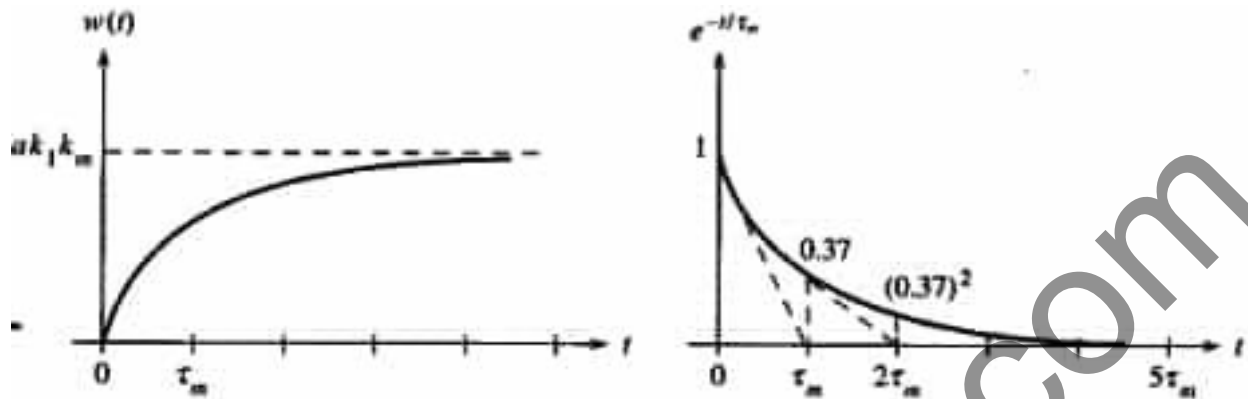
Here consider the armature-controlled dc motor driving a load, such as a videotape. The objective is to drive the tape at a constant speed. Note that it is an open-loop system.



$$G(s) = \frac{W(s)}{R(s)} = \frac{k_1 k_m}{\tau_m s + 1}; \text{ If } r(t) = au(t), W(s) = \frac{k_1 k_m}{\tau_m s + 1} \cdot \frac{a}{s} = \frac{ak_1 k_m}{s} - \frac{ak_1 k_m}{s + 1/\tau_m}$$

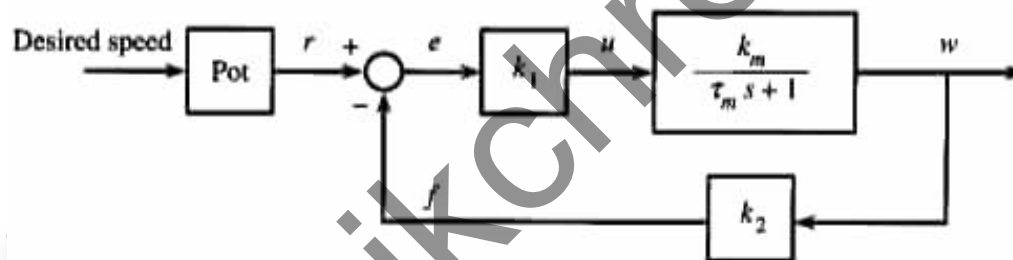
$$w(t) = ak_1 k_m - ak_1 k_m e^{-t/\tau_m}; \Rightarrow w_{ss}(t) = \lim_{t \rightarrow \infty} w(t) = ak_1 k_m$$

- $\omega_{ss}(t)$ is the steady-state final speed. If the desired speed is ω_r , choosing ' $a = \omega_r / k_1 k_m$ ' the motor will eventually reach the desired speed.



- From the time response e^{-t/τ_m} we concluded that for $t \geq 5\tau_m$ the value of e^{-t/τ_m} is less than 1% of its original value. Hence the speed of the motor will reach and stay within 1% of its final speed at 5-time constants.

Let us now consider the closed-loop system



$$\text{Here, } T(s) = \frac{W(s)}{R(s)} = \frac{k_1 k_m / (\tau_m s + 1)}{1 + k_1 k_2 k_m / (\tau_m s + 1)} = \frac{k_1 k_m}{\tau_m s + (1 + k_1 k_2 k_m)} = \frac{k_1 k_o}{\tau_o s + 1}$$

$$\text{where, } k_o = \frac{k_m}{1 + k_1 k_2 k_m} \text{ and } \tau_o = \frac{\tau_m}{1 + k_1 k_2 k_m}$$

If $r(t) = a$ then Response would be ; $w(t) = a k_1 k_o - a k_1 k_o e^{-t/\tau_o}$

If a is properly chosen, the tape can reach the desired speed. It will reach the desired speed in $5\tau_o$ seconds. Here $\tau_o = \tau_m$. So that we can control the speed of response in the feedback system.

Ramp response of first-order system

Let, $k_1 k_0 = 1$ for simplicity. Then, $T(s) = \frac{1}{\tau_o s + 1} = \frac{W(s)}{R(s)}$. Also, let, $r(t) = tu(t)$

$$\text{Then, } W(s) = \frac{1}{s^2(\tau_o s + 1)} = \frac{1}{s^2} - \frac{\tau_o}{s} + \frac{\tau_o^2}{\tau_o s + 1};$$

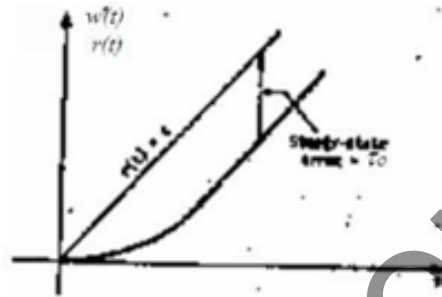
$$\Rightarrow w(t) = tu(t) - \tau_o(1 - e^{-t/\tau_o})u(t)$$

The error signal is, $e(t) = r(t) - w(t)$

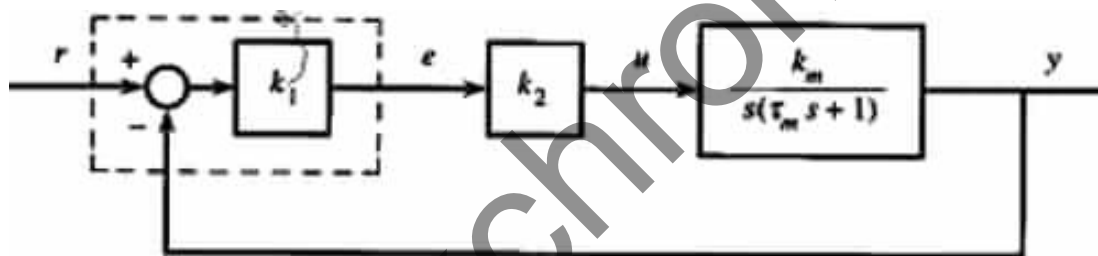
$$\text{Or, } e(t) = \tau_o(1 - e^{-t/\tau_o})u(t)$$

$$e_{ss}(t) = \tau_o$$

- Thus, the first-order system will track the unit ramp input with a steady-state error τ_o , which is equal to the time constant of the system.



Time-Response of Second-Order System



- Consider the antenna position control system. Its transfer function from r to y is,

$$T(s) = \frac{Y(s)}{R(s)} = \frac{k_1 k_2 k_m}{\tau_m s^2 + s + k_1 k_2 k_m} = \frac{k_1 k_2 k_m / \tau_m}{s^2 + \frac{1}{\tau_m} s + k_1 k_2 k_m / \tau_m} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

where we can define

$$(\omega_n)^2 = k_1 k_2 k_m / \tau_m; \text{ \& } 2\zeta \omega_n = 1 / \tau_m$$

The constant ξ is called the damping ratio and ω_n is called the natural frequency. The system above is, in fact, a standard second-order system.

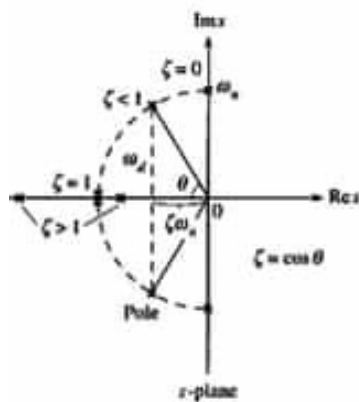
The transfer function $T(s)$ has two poles and no zero. Its poles are,

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

Natural frequency (ω_n): The natural frequency of a second-order system is the frequency of oscillation of the system without damping.

Damping ratio (ξ): The damping ratio is defined as the ratio of the damping factor σ , to the natural frequency ω_n .

Here, σ is called the damping factor, ω_d is called damped or actual frequency. The location of poles for different ξ are plotted in the given figure below. For $\xi=0$, the two poles $\pm j\omega_n$ are purely imaginary. If $0 < \xi < 1$, the two poles are complex conjugate.



Damping Ratio	Poles	Remark
$\zeta = 0$	Pure imaginary	Undamped
$0 < \zeta < 1$	Complex conjugate	Underdamped
$\zeta = 1$	Repeated real poles	Critically damped
$\zeta > 1$	Two distinct real poles	Overdamped

Unit Step Response of Second-Order System

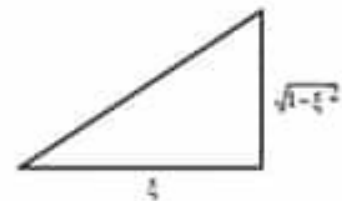
$$\text{Suppose, } r(t) = u(t), \Rightarrow R(s) = \frac{1}{s}, Y(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} \cdot \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Or,

$$Y(s) = \frac{1}{s} \cdot \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{1}{s} \cdot \frac{s + \zeta\omega_n + \zeta\omega_n}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}$$

Performing inverse Laplace transform,

$$y(t) = 1 - e^{-\zeta\omega_n t} \cos(\omega_n\sqrt{1-\zeta^2}t) - e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}t) \cdot \frac{\zeta}{\sqrt{1-\zeta^2}}$$



$$\text{or, } y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\sqrt{1-\zeta^2} \cos(\omega_n\sqrt{1-\zeta^2}t) + \zeta \sin(\omega_n\sqrt{1-\zeta^2}t) \right]$$

$$\text{or, } y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta), \text{ where, } \omega_d = \omega_n\sqrt{1-\zeta^2} \text{ and } \theta = \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right) = \cos^{-1} \zeta$$

$$y(t) = 1 - \frac{\omega_n}{\omega_d} e^{-\alpha t} \sin(\omega_d t + \theta)$$

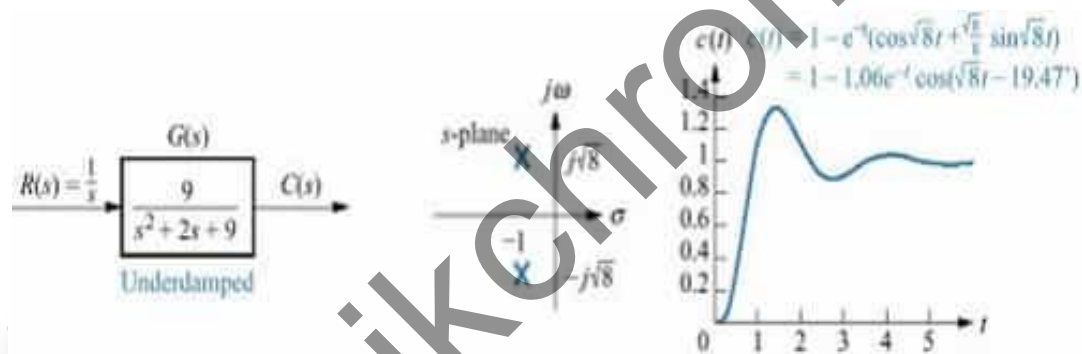
Second-Order Systems: General Specification

The second-order system exhibits a wide range of responses that must be analyzed and described. To become familiar with the wide range of responses before formalizing our discussion, we take a look at numerical examples of the second-order system responses shown in the figure.

- **Underdamped Response ($0 < \xi < 1$)**

This function has a pole at the origin that comes from the unit step and two complex poles that come from the system. The sinusoidal frequency is given the name of damped frequency of oscillation, ω_d . This response is shown in the figure called underdamped.

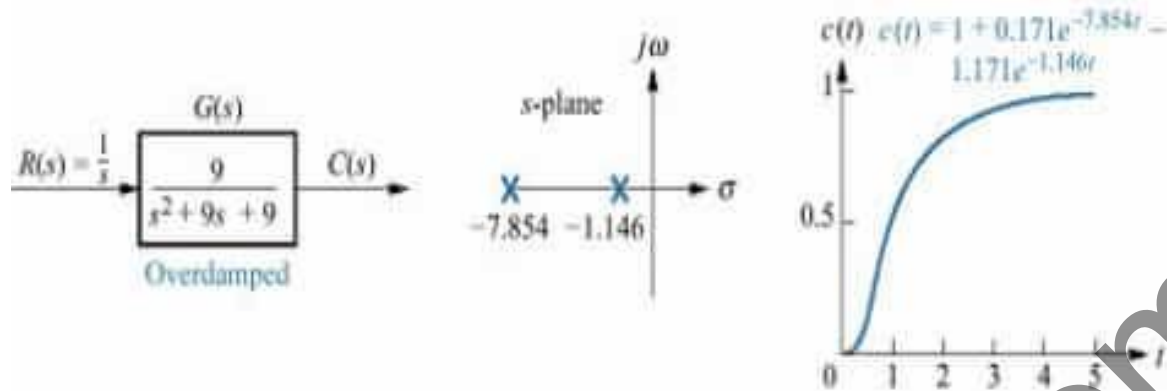
Example:



- **Overdamped System ($1 < \xi$)**

This function has a pole at the origin that comes from the unit step input and two real poles that come from the system. The input pole at the origin generates the constant forced response; each of the two system poles on the real axis generates an exponential natural frequency.

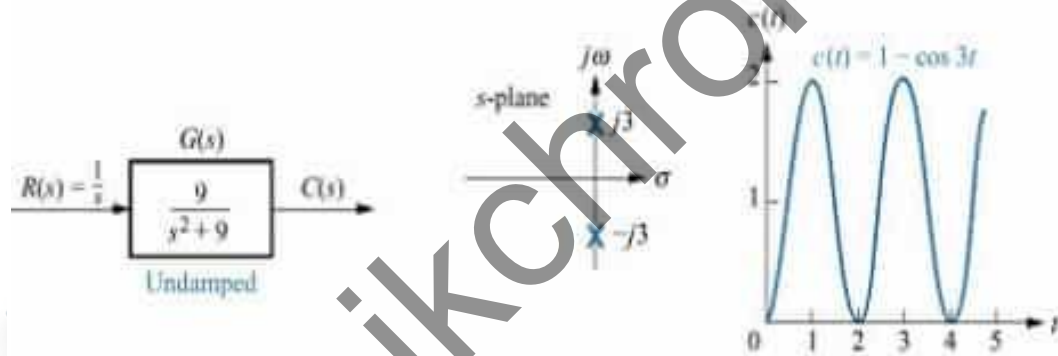
Example:



- **Undamped Response ($\xi = 0$)**

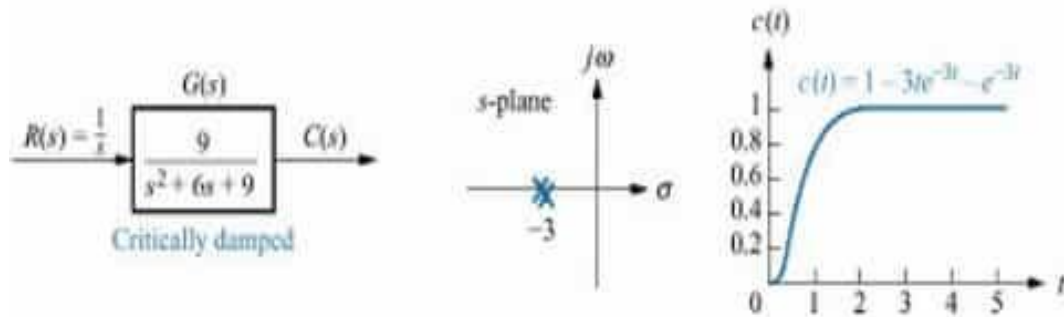
This function has a pole at the origin and two imaginary poles. The pole at the origin generates the constant forced response, and the two system poles on the imaginary axis at $\pm j3$ generate a Sinusoidal natural response.

Example:



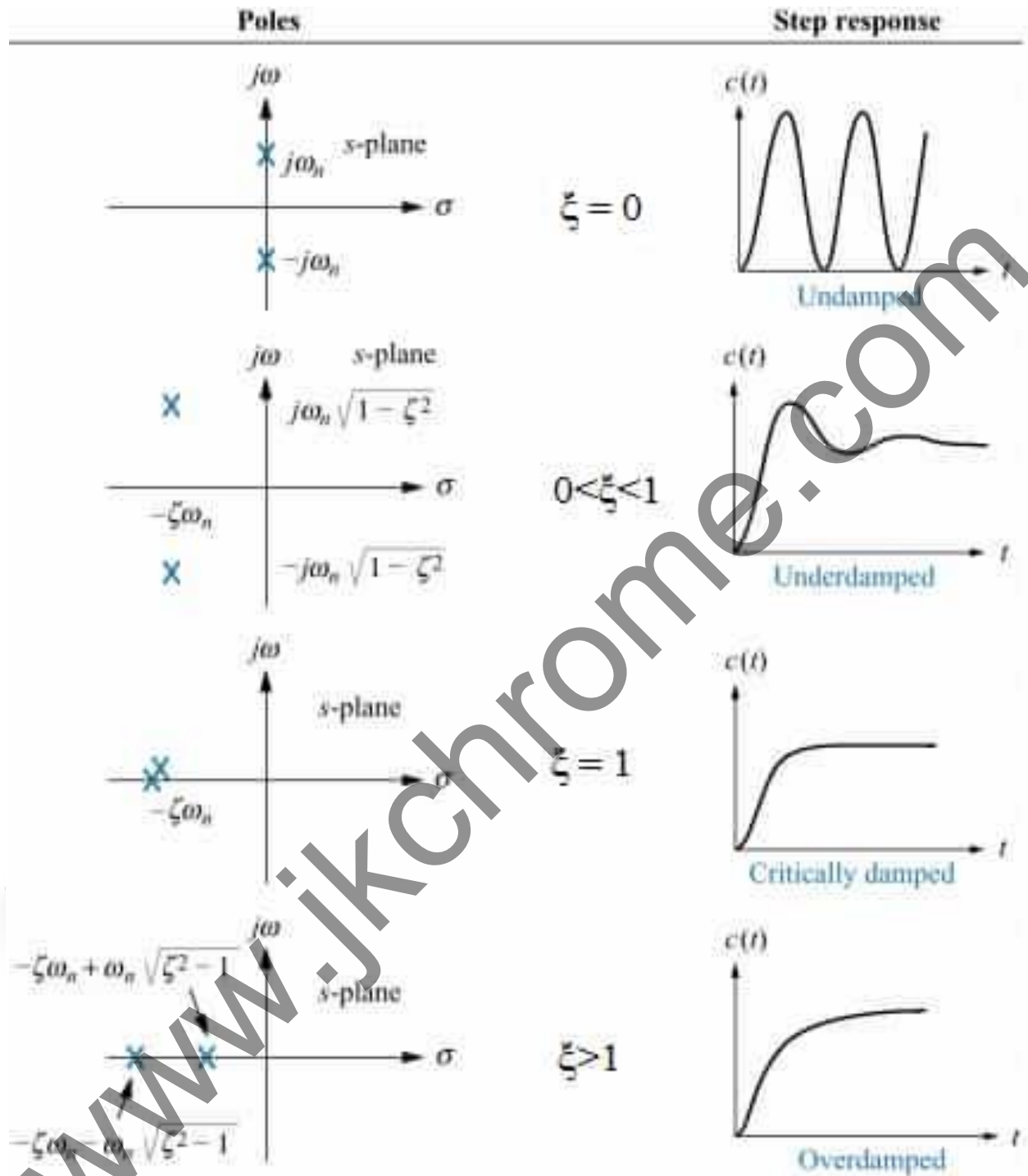
- **Critically Damped Response ($\xi = 1$)**

This function has a pole at the origin and two multiple real poles. The input pole at the origin generates the constant forced response, and two poles at the real axis at -3 generate a natural exponential response.



Note: In the above specifications of time domain, don't be confused with the number of Poles in $G(s)$, to Specify for which type of Damping is present for a particular case we consider the total number of poles are of transfer function i.e; $C(s)/R(s)$.

Summarization: Here once again we summarize the second-order damping functions as;



Time Domain Characteristics

In specifying the Transient-Response characteristics of a control system to a unit step input, we usually specify the following:

- **Delay time (t_d):** It is the time required for the response to reach 50% of the final value in the first attempt.
- **Rise time, (t_r):** It is the time required for the response to rise from 0 to 100% of the final value for the underdamped system.
- **Peak time, (t_p):** It is the time required for the response to reach the peak of time response or the peak overshoot.
- **Settling time, (t_s):** It is the time required for the response to reach and stay within a specified tolerance band (2% or 5%) of its final value.
- **Peak overshoot (M_p):** It is the normalized difference between the time response peak and the steady output and is defined as
- **Steady-state error (e_{ss}):** It indicates the error between the actual output and desired output as 't' tends to infinity.

Rise time, t_r : Put $y(t)=1$ at $t=t_r, \Rightarrow \sin(\omega_d t_r + \theta) = 0 = \sin \pi, \Rightarrow t_r = \frac{\pi - \theta}{\omega_d}; \theta = \cos^{-1} \zeta$

Peak time, t_p : Put $\frac{dy}{dt} = 0$ and solve for $t=t_p; 0 = \frac{\sigma \omega_n}{\omega_d} e^{-\sigma t} \sin(\omega_d t + \theta) - \omega_n e^{-\sigma t} \cos(\omega_d t + \theta)$

$$\Rightarrow \tan(\omega_d t_p + \theta) = \frac{\omega_d}{\sigma} = \frac{\omega_n \sqrt{1-\zeta^2}}{\zeta \omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan \theta, \Rightarrow \omega_d t_p = k\pi \quad k=0,1,2,\dots$$

Peak overshoot occurs at $k=1, \Rightarrow t_p = \pi / \omega_d = \pi / \omega_n \sqrt{1-\zeta^2}$.

Settling time, t_s : For 2% tolerance band, $\frac{\omega_n}{\omega_d} e^{-\sigma t_s} = 0.02, \Rightarrow t_s \cong \frac{4}{\sigma} = 4T$

Steady-state error e_{ss} : It is found previously that steady-state error for a step input is zero. Let us now consider **ramp input, $r(t) = tu(t)$** .

Then, $e_{ss} = \lim_{s \rightarrow 0} s \{R(s) - Y(s)\} = \lim_{s \rightarrow 0} s \left\{ \frac{1}{s^2} - \frac{1}{s^2} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right\}$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s} \left\{ 1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right\} = \lim_{s \rightarrow 0} \frac{1}{s} \left\{ \frac{s^2 + 2\zeta\omega_n s + \omega_n^2 - \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right\} = \frac{2\zeta\omega_n}{\omega_n^2} = \frac{2\zeta}{\omega_n}$$

- Therefore, the steady-state error due to ramp input is $2\zeta/\omega_n$.

Effect of Adding a Zero to a System:

If we add a zero at $s = -z$ be added to a second-order system. Then we have,

$$\frac{C(s)}{R(s)} = \frac{(s+z)\omega_n^2 / z}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{s}{z} \left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right)$$

- The multiplication term is adjusted to make the steady-state gain of the system unity.

Manipulation of the above equation gives,

- The effect of the added derivative term is to produce a pronounced early peak to the system response.
- The closer the zero to the origin, the more pronounced the peaking phenomenon.
- Due to this fact, the zeros on the real axis near the origin are generally avoided in design. However, in a sluggish system, the artful introduction of a zero at the proper position can improve the transient response.

Types of Feedback Control System:

The open-loop transfer function of a system can be written as

- If $n = 0$, the system is called type-0 system, if $n = 1$, the system is called type-1 system, if $n = 2$, the system is called type-2 system, etc.

Steady-State Error and Error Constants:

The steady-state performance of a stable control system is generally judged by its steady-state error to step, ramp, and parabolic inputs. For a unity feedback system,

$$E(s) = \frac{R(s)}{1+G(s)} \Rightarrow e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1+G(s)}$$

It is seen that steady-state error depends upon the input $R(s)$ and the forward transfer function $G(s)$.

For unit-step input: $r(t) = u(t)$, $R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{z \rightarrow 0} zE(s) = \lim_{z \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+G(0)} = \frac{1}{1+k_p}; k_p \text{ is called position error constant.}$$

For unit-ramp input: $r(t) = tu(t)$, $R(s) = \frac{1}{s^2}$

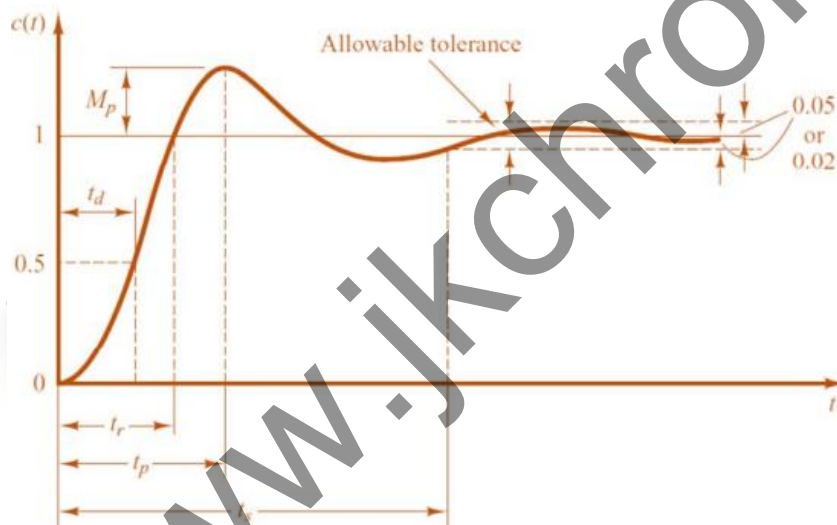
$$e_{ss} = \lim_{z \rightarrow 0} zE(s) = \lim_{z \rightarrow 0} \frac{1}{s[1+G(s)]} = \lim_{z \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{k_v}; k_v \text{ is called velocity error constant.}$$

For unit-parabolic input: $r(t) = t^2 / 2$, $R(s) = \frac{1}{s^3}$

$$e_{ss} = \lim_{z \rightarrow 0} zE(s) = \lim_{z \rightarrow 0} \frac{1}{s^2[1+G(s)]} = \lim_{z \rightarrow 0} \frac{1}{s^2G(s)} = \frac{1}{k_a}; k_a \text{ is called acceleration error constant.}$$

Time Domain Characteristics

In specifying the Transient-Response characteristics of a control system to a unit step input, we usually specify the following:



- **Delay time (t_d):** It is the time required for the response to reach 50% of the final value in the first attempt.

The expression of delay time, t_d for second-order system is:

$$t_d = \frac{1 + 0.7\xi}{\omega_n}$$

- **Rise time, (t_r):** It is the time required for the response to rise from 0 to 100% of the final value for the under-damped system.

The expression of rise time, t_r for second-order system is:

$$t = \frac{2\pi - \phi}{\omega_d}$$

- **Peak time, (t_p):** It is the time required for the response to reach the peak of time response or the peak overshoot.

The expression of peak time, t_p for second-order system is:

$$t_p = \frac{n\pi}{\omega_d} \text{ seconds}$$

For first peak, $n = 1$ (maxima)

$$t_p = \frac{\pi}{\omega_d}$$

For first minima, $n = 2$

$$t_p = \frac{2\pi}{\omega_d}$$

For second maxima, $n = 3$

$$t_p = \frac{3\pi}{\omega_d}$$

- **Settling time, (t_s):** It is the time required for the response to reach and stay within a specified tolerance band (2% or 5%) of its final value.

The expression of settling time, t_s for second-order system is:

For 2% tolerance band,

$$t_s = \frac{4}{\xi\omega_n}$$

For 5% tolerance band,

$$t_s = \frac{3}{\xi\omega_n}$$

- **Peak overshoot (M_p):** It is the normalized difference between the time response peak and the steady output and is defined as

$$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The expression of peak overshoot, M_p for second-order system is:

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

or

$$M_p = e^{-\pi \cot \phi}$$

or

$$M_p = e^{-\frac{\pi}{\tan \phi}}$$

$$\text{Where } \tan \phi = \frac{\sqrt{1-\xi^2}}{\xi}$$

- **Steady-state error (e_{ss}):** It indicates the error between the actual output and desired output as 't' tends to infinity.

$$e_{ss} = \lim_{t \rightarrow \infty} [r(t) - c(t)].$$

Effect of Adding a Zero to a System:

If we add a zero at $s = -z$ be added to a second-order system. Then we have,

$$\frac{C(s)}{R(s)} = \frac{(s+z)\omega_n^2/z}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

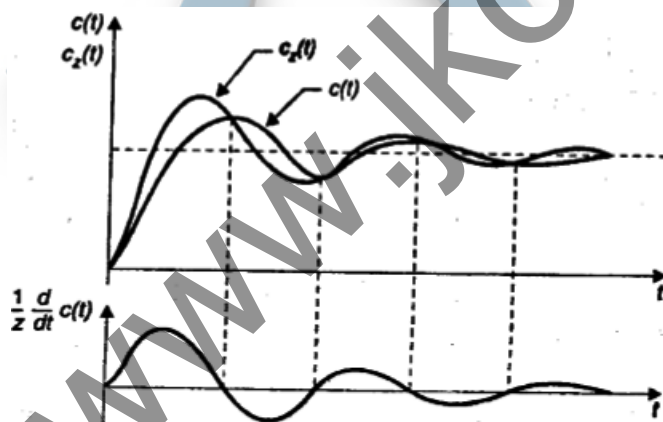
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} + \frac{s}{z} \left(\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \right)$$

- The multiplication term is adjusted to make the steady-state gain of the system unity.

Manipulation of the above equation gives,

$$c_x(t) = c(t) + \frac{1}{z} \frac{d}{dt} c(t).$$

- The effect of the added derivative term is to produce a pronounced early peak to the system response.
- The closer the zero to the origin, the more pronounced the peaking phenomenon.
- Due to this fact, the zeros on the real axis near the origin are generally avoided in design. However, in a sluggish system, the artful introduction of a zero at the proper position can improve the transient response.



Types of Feedback Control System:

The open-loop transfer function of a system can be written as

$$G(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots}{s^n(s+p_1)(s+p_2)(s+p_3)\cdots} = \frac{K'(T_{z1}s+1)(T_{z2}s+1)(T_{z3}s+1)\cdots}{s^n(T_{p1}s+1)(T_{p2}s+1)(T_{p3}s+1)\cdots}$$

- If $n = 0$, the system is called type-0 system, if $n = 1$, the system is called type-1 system, if $n = 2$, the system is called type-2 system, etc.

Steady-State Error and Error Constants:

The steady-state performance of a stable control system is generally judged by its steady-state error to step, ramp and parabolic inputs. For a unity feedback system,

$$E(s) = \frac{1}{1 + G(s) \cdot H(s)} R(s)$$

Where,

$E(s)$ is error signal

$R(s)$ is input signal

$G(s) H(s)$ is the open loop transfer function

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ s \left[\frac{1}{1 + G(s)H(s)} \right] R(s) \right\}$$

It is seen that steady-state error depends upon the input $R(s)$ and the forward transfer function $G(s)$.

1. If input is unit step i.e $R(t) = u(t)$

$$R(s) = \frac{1}{s}$$

Steady state error is

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ s \left[\frac{1}{1+G(s)H(s)} \right] R(s) \right\}$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ s \left[\frac{1}{1+G(s)H(s)} \right] \frac{1}{s} \right\}$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ \left[\frac{1}{1+G(s)H(s)} \right] \right\}$$

$$e_{ss} = \left[\frac{1}{1 + \lim_{s \rightarrow 0} G(s)H(s)} \right]$$

$$e_{ss} = \frac{1}{1 + K_p}$$

Where

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

K_p is positional error constant.

2. If input is unit ramp i.e $R(t) = tu(t)$

$$R(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ s \left[\frac{1}{1+G(s)H(s)} \right] R(s) \right\}$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ s \left[\frac{1}{1+G(s)H(s)} \right] \frac{1}{s^2} \right\}$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ \left[\frac{1}{s(1+G(s)H(s))} \right] \right\}$$

$$e_{ss} = \left[\frac{1}{0 + \lim_{s \rightarrow 0} sG(s)H(s)} \right] = \left[\frac{1}{\lim_{s \rightarrow 0} sG(s)H(s)} \right]$$

$$e_{ss} = \frac{1}{K_v}$$

Where

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

K_v is velocity error constant.

3. If input is unit parabolic i.e $R(t) = 0.5tu(t)$

$$R(s) = \frac{1}{s^3}$$

Steady state error,

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ s \left[\frac{1}{1+G(s)H(s)} \right] R(s) \right\}$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ s \left[\frac{1}{1+G(s)H(s)} \right] \frac{1}{s^3} \right\}$$

$$e_{ss} = \lim_{s \rightarrow 0} \left\{ \left[\frac{1}{s^2(1+G(s)H(s))} \right] \right\}$$

$$e_{ss} = \left[\frac{1}{s^2 + \lim_{s \rightarrow 0} s^2 G(s)H(s)} \right] = \left[\frac{1}{0 + \lim_{s \rightarrow 0} s^2 G(s)H(s)} \right]$$

$$e_{ss} = \frac{1}{K_a}$$

Where

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$$

K_a is acceleration error constant.

Frequency Domain Analysis

In a feedback control system, at least part of the information used to change the output variable is derived from measurements performed on the output variable itself. This type of closed-loop control is often used in preference to open-loop control (where the system does not use output-variable information to influence its output) since feedback can reduce the sensitivity of the system to externally applied disturbances and to changes in system parameters.

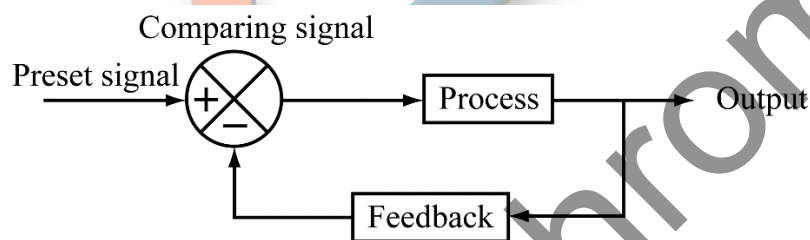
Familiar examples of feedback control systems include residential heating systems, most high-fidelity audio amplifiers, and the iris-retina combination that regulates light entering the eye.

Closed-loop control system:

- Sometimes, we may use the output of the control system to adjust the input signal. This is called feedback. Feedback is a special feature of a closed-loop control system.
- A closed-loop control system compares the output with the expected result or command status, then it takes appropriate control actions to adjust the input signal.
- Therefore, a closed-loop system is always equipped with a sensor, which is used to monitor the output and compare it with the expected result.

The following figure shows a simple closed-loop system.

- The output signal is feedback to the input to produce a new output.
- A well-designed feedback system can often increase the accuracy of the output.



Block diagram of a closed-loop control system

Feedback can be divided into positive feedback and negative feedback.

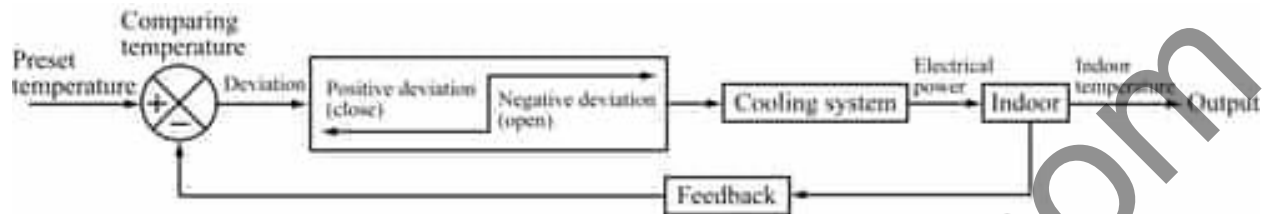
Positive Feedback:

- Positive feedback causes the new output to deviate from the present command status.
- For example, an amplifier is put next to a microphone, so the input volume will keep increasing, resulting in a very high output volume.

Negative Feedback:

- Negative feedback directs the new output towards the present command status, so as to allow more sophisticated control.
- For example, a driver has to steer continuously to keep his car on the right track.

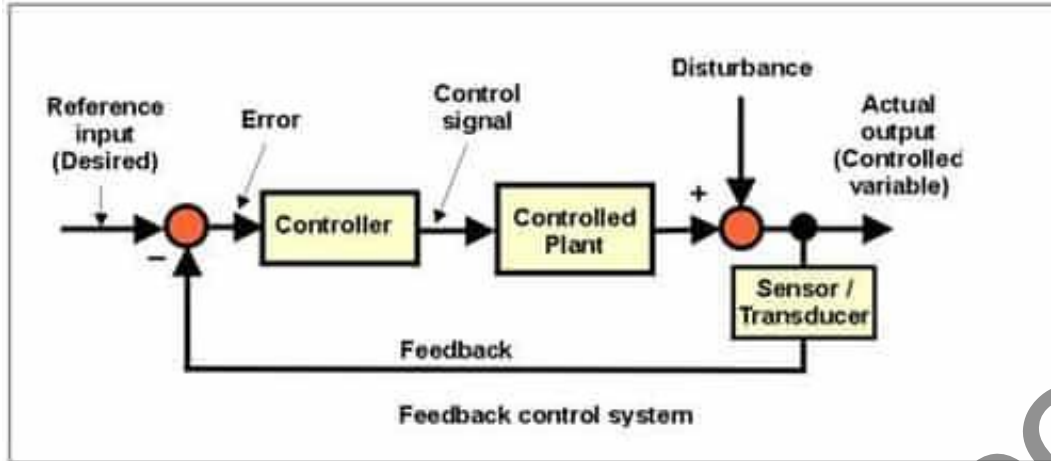
Most modern appliances and machinery are equipped with closed-loop control systems. Examples include air conditioners, refrigerators, automatic rice cookers, automatic ticketing machines, etc. An air conditioner, for example, uses a thermostat to detect the temperature and control the operation of its electrical parts to keep the room temperature at a preset constant.



Block diagram of the control system of an air conditioner

- One advantage of using the closed-loop control system is that it is able to adjust its output automatically by feeding the output signal back to the input.
- When the load changes, the error signals generated by the system will adjust the output. However, closed-loop control systems are generally more complicated and thus more expensive to make.
- Feedback is a common and powerful tool when designing a control system.
- The feedback loop is the tool that takes the system output into consideration and enables the system to adjust its performance to meet a desired result of the system.

In any control system, the output is affected due to a change in environmental conditions or any kind of disturbance. So one signal is taken from the output and is fed back to the input. This signal is compared with reference input and then an error signal is generated. This error signal is applied to the controller and output is corrected. Such a system is called a feedback system.



- When the feedback signal is positive then the system called a positive feedback system. For a positive feedback system, the error signal is the addition of reference input signal and the feedback signal.
- When the feedback signal is negative then the system is called a negative feedback system. For a negative feedback system, the error signal is given by the difference of reference input signal and the feedback signal.

Feedback characteristics: Including feedback into the control of a system results in the following.

Advantages:

- Increased accuracy. The output can be made to reproduce the input
- Reduced sensitivity to system characteristics
- Reduction in the effect of non-linearities
- Increased bandwidth. The system can be made to respond to a larger range of input frequencies.

The major disadvantages resulting from feedback are the increased risk of instability and the additional cost of design and implementation.

Applications of Feedback: Flight control systems, Robotics, Chemical process control, Communications & Networks, Automotive, Biological Systems, Environmental Systems & Quantum Systems.

Rule to Draw the Root Locus Plot

Rule 1-Symmetry: Since the characteristic equation has real coefficients, any zeros must occur in complex conjugate pairs (which are symmetric about the

real axis). Since the root locus is just a diagram of the roots of the characteristic equation as K varies, it must also be symmetric about the real axis.

Rule 2-Number of Branches: Since the order of the characteristic equation is the same as that of the denominator of the loop gain, the number of branches is n , the order of the denominator polynomial.

Rule 3- Starting and Ending Points: Start from the magnitude condition: $K|N(s)/D(s)| = 1$

- So the locus **starts (when $K=0$) at poles** of the loop gain, and **ends (when $K \rightarrow \infty$) at the zeros**. Note: there are q zeros of the loop gain as $s \rightarrow \infty$.

Rule 4- Locus On The Real Axis: The locus exists **on the real axis to the left** of a **sum of the number of poles and zeros is odd** on the axis.

Rule 5- Number of Branches Terminating to Infinity: If for a given system the number of **Open Loop Poles** are P & the number of **Open Loop Zeros** are Z then there will be " $P-Z$ " branches which are terminating to **Infinity**.

Rule 6- Asymptotes Angle & Centroid: We know that if we have a characteristic equation $P(s)$ that has more poles ($P=N$) than zeros ($Z=M$) then " $N-M$ " of the root locus branches tend to zeros at infinity.

These asymptotes intercept the real axis at a point, Called it **Centroid σ_A** ,

$$\sigma_A = \frac{\sum_{j=1}^N p_j - \sum_{i=1}^M z_i}{N - M}$$

Or in other words

$$\sigma_A = \frac{\sum(\text{finite poles}) - \sum(\text{finite zeros})}{N - M}$$

- The angles of the asymptotes ϕ_k are given by for $K > 0$

$$\phi_k = \frac{2q + 1}{P - Z} (180^\circ)$$

where, $q = 0, 1, 2, \dots, n - m - 1$.

- The angles of the asymptotes ϕ_k are given by for $K < 0$

$$\phi_k = \frac{2q}{P - Z} (180^\circ)$$

where, $q = 0, 1, 2, \dots, n - m - 1$.

Rule 7-Determining the Breakaway Points:

- First of all, we have to identify the portions of the real axis where a breakaway point must exist.
- Assuming we have already marked the segments of the real axis that are on the root locus, we need to find the segments that are the part of the root locus by either two poles or two zeros (either finite zeros or zeros at infinity).
- To estimate the values of s at the breakaway points, the characteristic equation \Rightarrow
- $1 + KP(s) = 0$ is rewritten in terms of K as

$$K = -\frac{1}{P(s)} = K(s)$$

To find the breakaway points we find the values of s corresponding to the maxima in $K(s)$. i.e. where $dK/ds = K'(s) = 0$.

- As the last step, we check the roots of $K'(s)$ that lie on the real axis segments of the locus. The roots that lie in these intervals are the breakaway points. ♦

Rule 8- Find the angles of departure/arrival for complex poles/zeros:

- The angle of departure (θ_d) is given by $\theta_d = 180^\circ + \arg [G(s)H(s)]$ where $\arg [G(s)H(s)]$ is the angle of $G(s)H(s)$ excluding the pole where the angle is to be calculated.
- Similarly, the angle of arrival is given by $\theta_a = 180^\circ - \arg [G(s)H(s)]$, where $\arg [G(s)H(s)]$ is the angle of $G(s)H(s)$ excluding the zero, where the angle is to be calculated.
- When there are complex poles or zeros of $P(s)$, the root locus branches will either depart or arrive at an angle θ where, for a complex pole or zero at $s = p$ or $s = z$,

$$\angle P^*(p) - \theta = \pi \text{ for } s = p$$

$$\angle P^*(z) + \theta = \pi \text{ for } s = z$$

where

$$P^*(p) = \frac{\prod_{i=1}^M (p - z_i)}{\prod_{\substack{j=1 \\ p \neq p_j}}^N (p - p_j)}$$

$$P^*(z) = \frac{\prod_{\substack{i=1 \\ z \neq z_i}}^M (z - z_i)}{\prod_{j=1}^N (z - p_j)}$$

In other words

$$P^*(p) = (s - p)P(s) \Big|_{s=p}$$

$$P^*(z) = \frac{P(s)}{s - z} \Big|_{s=z}$$

Therefore, exploiting the rules of complex numbers, we can rewrite $\angle P^*(p)$ and $\angle P^*(z)$ as

$$\angle P^*(p) = \sum_{i=1}^M \angle(p - z_i) - \sum_{\substack{j=1 \\ p \neq p_j}}^N \angle(p - p_j)$$

$$\angle P^*(z) = \sum_{\substack{i=1 \\ z \neq z_i}}^M \angle(z - z_i) - \sum_{j=1}^N \angle(z - p_j)$$

Rule 9- Intersection of the Root Locus with the $j\omega$ Axis: To find the intersection of root locus with the imaginary axis. The following procedures are followed

Step 1: Construct the characteristic equation $1 + G(s)H(s) = 0$

Step 2: Develop Routh array in terms of K.

Step 3: Find K_{mar} that creates one of the roots of the Routh array as a row of zeros.

Step 4: Frame auxiliary equation $A(s) = 0$ with the help of the coefficient of a row just above the row of zeros.

Step 5: The roots of the auxiliary equation $A(s) = 0$ for $K = K_{\text{mar}}$ give the intersection points of the root locus with the imaginary axis.

Value Gain Margin:

Gain Margin (GM) =

$$\frac{\text{Value of } K \text{ at imaginary crossover}}{\text{Design value of } K}$$

Note: If the root locus does not cross the $j\omega$ axis, the gain is ∞ . GM represents the maximum gain that can be multiplied for the system to be just on the verge of instability.

Phase Margin: Phase margin (PM) can be determined for a given value of K as follows

- Calculate ω for which $|G(j\omega)H(j\omega)| = 1$ for the given value of K .
- Calculate $[G(j\omega)H(j\omega)]$
- Phase margin = $180^\circ + \arg [G(j\omega)H(j\omega)]$

Routh-Hurwitz Stability Criterion

The technique **Routh-Hurwitz criterion** is a method to know whether a linear system is stable or not by examining the locations of the roots of the characteristic equation. The method determines only if there are roots that lie outside of the left half-plane; while it does not actually compute the roots.

Here we have a Transfer Function $F(s)$:

$$F(s) = \frac{b_0s^m + b_1s^{m-1} + \dots + b_{m-1}s + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n} = N(s)/D(s)$$

where $N(s)$ & $D(s)$ are the Polynomials in s to know the Poles (by equating $D(s)=0$) & Zeros (by equating $N(s)=0$).

The Characteristic Equation of the above Transfer Function can be written as

$$1+G(s)H(s) = 0 ; \text{ or } D(s)= a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

To find out whether the system is stable or not, check the following conditions:

1. Two necessary but not sufficient conditions that all the roots have negative real parts are

- All the polynomial coefficients must have the same sign.
- All the polynomial coefficients must be non-zero.

2. If condition (1) is satisfied, then compute the Routh-Hurwitz array as follows

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \dots \\ s^{n-2} & b_1 & b_2 & b_3 & & \dots \\ s^{n-3} & c_1 & c_2 & c_3 & & \dots \\ s^{n-4} & & & \vdots & & \\ \vdots & & & \vdots & & \\ s^1 & & & \vdots & & \\ s^0 & & & \vdots & & \end{array}$$

- where the a_i are the polynomial coefficients, these coefficients in the rest of the table are computed as follow:

$$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{-1}{a_{n-1}} (a_n a_{n-3} - a_{n-2} a_{n-1}) \quad b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$b_3 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix} \dots \quad c_1 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \quad c_2 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}$$

- The necessary condition that all roots have negative real parts is that all the elements of the first column of the array have the same sign.
- The number of changes of sign equals the number of roots with positive real parts.

Special Case-1: Zero First-Column Element

- If the **first element of a row is zero**, but some **other** elements in that row are **nonzero**. Here we can replace the zero elements with " ϵ ", & complete

the table, and then interpret the results assuming that " ϵ " is a small number of the same sign as the element above it. The results must be interpreted in the limit as $\epsilon \rightarrow 0$

Special Case-2: If Complete Zero Row.

- Whenever **all the coefficients** in a row are zero, a pair of roots of equal magnitude and the opposite sign will be present.
- Here the Possibilities of these **two roots could be Real or Conjugate Imaginary** with equal magnitudes and opposite signs.
- The zero row is replaced by taking the coefficients of $dP(s)/ds$, where $P(s)$, called the auxiliary polynomial, is obtained from the values in the row above the zero row. The pair of roots can be found by solving $dP(s)/ds = 0$.
- It is to be noted that the **auxiliary polynomial** always has **even degree**.

Example: Use of Auxiliary Polynomial

Consider the quintic equation $A(s) = 0$ where $A(s)$ is

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 50 = 0$$

Solution: The Routh array starts off as

$$\begin{array}{cccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \leftarrow \text{auxiliary polynomial } P(s) \\ s^3 & 0 & 0 & \end{array}$$

The auxiliary polynomial $P(s)$ will be

$$P(s) = 2s^4 + 48s^2 - 50$$

which shows that $A(s) = 0$ must have two pairs of roots of equal magnitude and opposite sign, which are also roots of the auxiliary polynomial equation $P(s) = 0$.

Considering derivative of $P(s)$ with respect to s we get $\Rightarrow dP(s)/ds = 8s^3 + 96s$.

so the s^3 row is as shown below and the Routh array will be

s^5	1	24	-25	
s^4	2	48	-50	
s^3	8	96		← Coefficients of $dP(s)/ds$
s^2	24	-50		
s^1	112.7	0		
s^0	-50			

- Now there is a single change of sign in the first column of the resulting array, indicating that there $A(s) = 0$ has one root with a positive real part.

On Solving the Auxiliary Polynomial $2s^4 + 48s^2 - 50 = 0$ it yields the remaining roots such as

- $s^2 = 1 \Rightarrow s = \pm 1$
- $s^2 = -25 \Rightarrow s = \pm j5$
- so the original equation can be factored as $\Rightarrow (s + 1)(s - 1)(s + j5)(s - j5)(s + 2) = 0$.

Relative Stability Analysis

- The **Routh's stability criterion** provides the answer to the question of **absolute stability**. This, in many practical cases, is not sufficient. We usually require information about the relative stability of the system.
- A real approach to examine relative stability is to shift the s-plane, here we substitute $s = z - \sigma$ ($\sigma = \text{constant}$) into the characteristic equation, write the polynomial in terms of z and then apply Routh's stability criterion to the new polynomial in z.
- The number of changes of sign in the first column of the array of Polynomial in z is equal to the number of roots which are located to the right of the vertical line $s = -\sigma$. Thus, this test reveals the number of roots which lie to the right of the vertical line $s = -\sigma$.

Jury Stability Test:

According to Jury stability test:

if $P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$, where $a_0 > 0$.

Then

Row	z^0	z^1	z^2	z^3	z^4	...	z^n
1	a_n	a_{n-1}	a_{n-2}	a_0
2	a_0	a_1	a_2	a_n
3	b_{n-1}	b_{n-2}	b_0		
4	b_0	b_1	b_{n-1}	
5	c_{n-2}	c_{n-3}	...	c_0			
6	c_0	c_1	c_{n-2}		

Where

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \quad k = 0, 1, 2, 3, \dots, n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \quad k = 0, 1, 2, 3, \dots, n-2$$

This system will be stable if

- $|a_n| < a_0$
 - $P(z)|_{z=1} > 0$
 - $P(z)|_{z=-1} > 0$ for n even and $P(z)|_{z=-1} < 0$ for n odd
- $$\begin{aligned} |b_{n-1}| &> |b_0| \\ |c_{n-2}| &> |c_0| \\ &\dots \\ &\dots \\ |q_2| &> |q_0| \end{aligned}$$

Example: Verify Jury Stability Test for the given Polynomial.

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

$$\text{Thus, } a_0 = 1 \quad a_1 = -1.2 \quad a_2 = 0.07 \quad a_3 = 0.3 \quad a_4 = -0.08$$

We will now check the stability conditions.

- $|a_n| = |a_4| = 0.08 < a_0 = 1 \Rightarrow$ First condition is satisfied.
- $P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0 \Rightarrow$ Second condition is satisfied.
- $P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0 \Rightarrow$ Third condition is satisfied.

hence the system is stable.

Various Plots

Bode Plot

The bode plot gives a graphical method for determining the stability of a control system based on sinusoidal frequency response. Bode graphs are representations of the magnitude and phase of $G(j\omega)$ (where the frequency vector ω contains only positive frequencies).

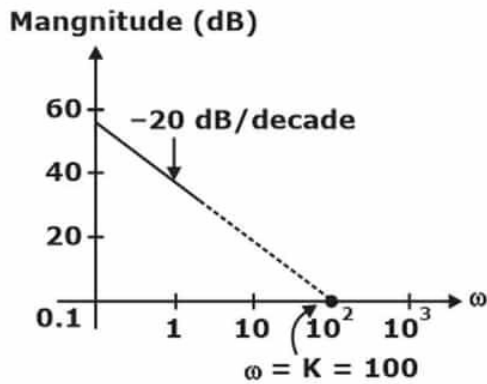
The bode plot consists of two graphs: Magnitude plot and Phase plot

- $20 \log_{10} |G(j\omega)|$ versus $\log \omega$, this is called the magnitude plot.
- Phase shift (in degrees) versus $\log \omega$ (frequency), is called phase plot.
- Bode plots are asymptotic log magnitude and phase plots. These are drawn as straight lines.
- The corner frequency is the frequency at which the slope of the asymptotic log magnitude plot changes.
- The frequency band from ω_1 and ω_2 such that $(\omega_2/\omega_1) = 10$ is called a decade.
- For a first-order factor, the slope of the Log magnitude plot changes by ± 20 dB/decade at the corner frequency according to as the factor is in the numerator or denominator respectively. For a second-order factor, the slope changes by ± 40 dB/decade and so on.

Initial Slope of the Bode Plot: Initial slope can be determined by the type of the system, Its value is different for different types of system.

- **Type 0 system:** For this system, the initial slope is 0 dB/decade with dB value $20 \log K$.
- **Type 1 system:** For this system, the initial part of the system is

$$20 \log |G(j\omega) H(j\omega)| = 20 \log K - 20 \log \omega$$

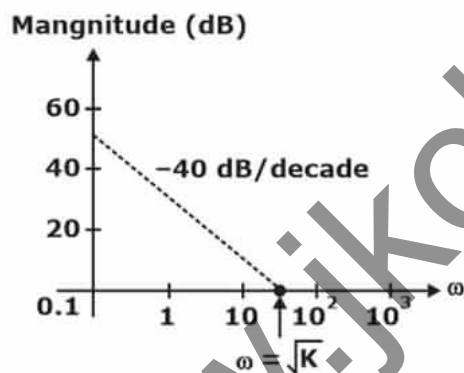


Intial part of plot for type 1 system

- **Type 2 system:** For type 2 system, the initial part of the plot is

$$20\log_{10}|G(j\omega) H(j\omega)| = 20\log K - 40\log\omega$$

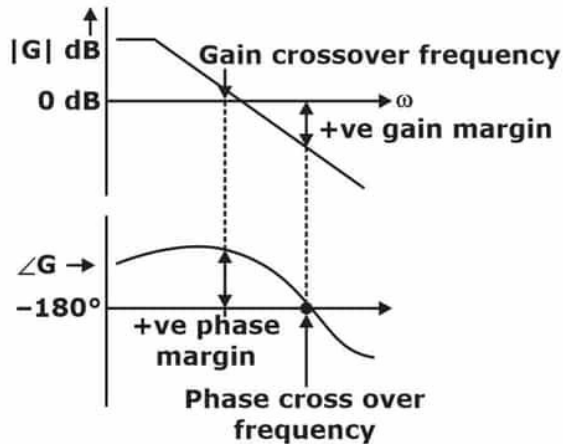
Here, the initial slope is -40 dB/decade and plot cuts the 0 dB line at $\omega = \sqrt{K}$



Intial part of bode plot for type 2 system

Phase Margin and Gain Margin

The PM and GM can be calculated by considering the following plot



- **Gain Crossover Frequency:** Frequency at which magnitude plot crosses the 0 dB line.
- **Phase Crossover Frequency:** Frequency at which phase plot crosses the -180° line
- **Gain Margin:** Value of gain from gain plot at phase crossover frequency is called gain margin. Gain margin is positive if it is below zero dB line
- **Phase Margin:** Value of phase from phase plot at the gain crossover frequency is known as phase margin. Phase margin is positive if it is above -180° line

Minimum Phase, Non-minimum Phase and All Pass Transfer Function

- Minimum phase transfer function:

$$G_1(s) = \frac{1 + sT_1}{1 + sT_2}$$

- Non-minimum phase system:

$$G_2(s) = \frac{1 - sT_1}{1 + sT_2}$$

- All pass transfer functions:

$$G_3(s) = \frac{1 - sT}{1 + sT}$$

- If there are no poles of the transfer function at the origin, the initial slope of the LM plot is zero and the magnitude is $20 \log K_p$ up to the lowest corner frequency.
- If there is a single pole at the origin of the open loop transfer function the LM plot has an initial slope equal to -20 dB/decade up to the lowest corner frequency and if this line is extended, it will intersect the frequency axis at $\omega = k_v$.
- If there are two poles at the origin of the open loop transfer function, the LM plot has an initial slope equal to -40 dB/decade up to the lowest corner frequency and this line is extended will intersect the frequency axis at $\omega = \sqrt{K_a}$.
- If there is one zero at the origin, the LM plot has an initial slope equal to 20 dB/decade up to the lowest corner frequency. If there are two zeros at the origin, the LM plot has an initial slope equal to $+40 \text{ dB/decade}$ up to the lowest corner frequency and so on
- Multiplication of the transfer function by a gain factor is equivalent to shifting the LM plot vertically up by an equivalent gain in dB.

Effects of Addition of Poles: The effects of addition of poles are as follows

- There is change in shape of the root locus and it shifts towards the imaginary axis. The intercept on the $j\omega$ axis occurs for a lower value of K because of asymptote angle being lower down.
- System becomes oscillatory.
- Gain margin and relative stability decrease.
- There is reduction in the range of K .
- A sluggish response can be changed to a quicker response for the artful introduction of a pole.
- Settling time increases.

Effects of Addition of Zeros: The effects of addition of zeros are as follows

- There is change in shape of the root locus and it shifts towards the left of the s -plane.
- Stability of the system is enhanced.
- Range of K increases.
- Settling time speeds up.

Polar Plot

The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω varied from zero to infinity.

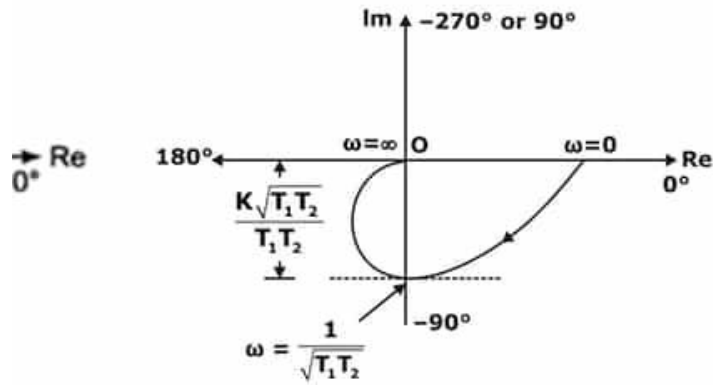
Procedure to Sketch the Polar Plot:

- **Step-1:** Determine transfer function $G(j\omega)$.
- **Step-2:** Put $s = j\omega$ in the transfer function to obtain $G(j\omega)$.
- **Step-3:** At $\omega = 0$ and $\omega = \infty$, calculate $|G(j\omega)|$
by $\lim_{\omega \rightarrow 0} |G(j\omega)|$ and $\lim_{\omega \rightarrow \infty} |G(j\omega)|$.
- **Step-4:** Calculate the phase angle of $G(j\omega)$ at $\omega = 0$ and $\omega = \infty$
by $\lim_{\omega \rightarrow 0} \angle G(j\omega)$ and $\lim_{\omega \rightarrow \infty} \angle G(j\omega)$.
- **Step-5:** Rationalize the function $G(j\omega)$ and separate the real and imaginary axis.
- **Step-6:** Equate the imaginary part $\text{Im}|G(j\omega)|$ to zero and determine the frequencies at which plot intersects the real axis and calculate the value of $G(j\omega)$ at the point of intersection by substituting the determined value of frequency in the expression of $G(j\omega)$.
- **Step-7:** Equate the real part $\text{Re}|G(j\omega)|$ to zero and determine the frequencies at which plots intersects the imaginary axis and calculate the value $G(j\omega)$ at the point of intersection by substituting the determined value of frequency in the rationalized expression of $G(j\omega)$.
- **Step-8:** Sketch the polar plot with the help of the above information.

Polar Plot of Some Standard Functions:

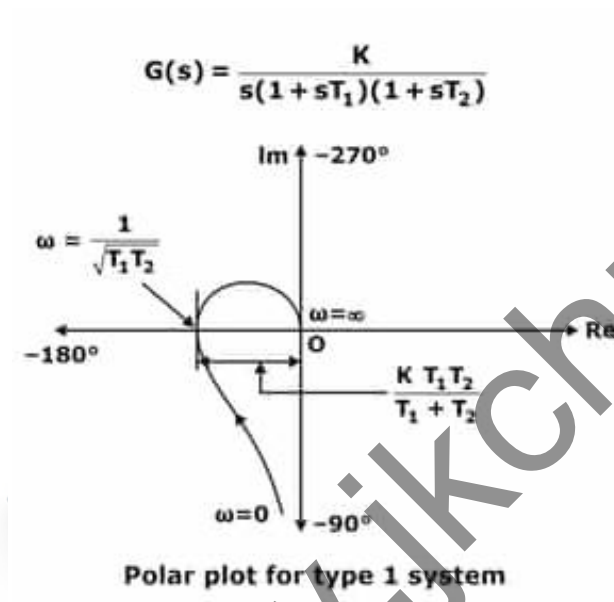
- **Type 0 System**

$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$



Polar plot for type 0 system

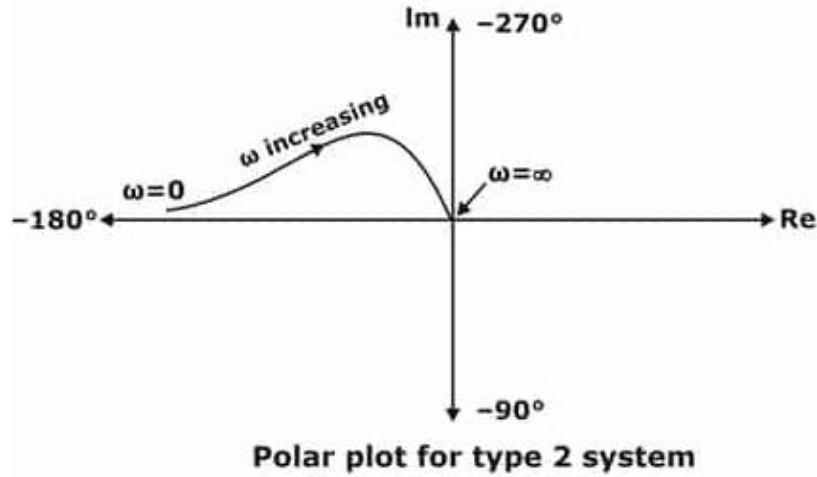
- Type 1 System



Polar plot for type 1 system

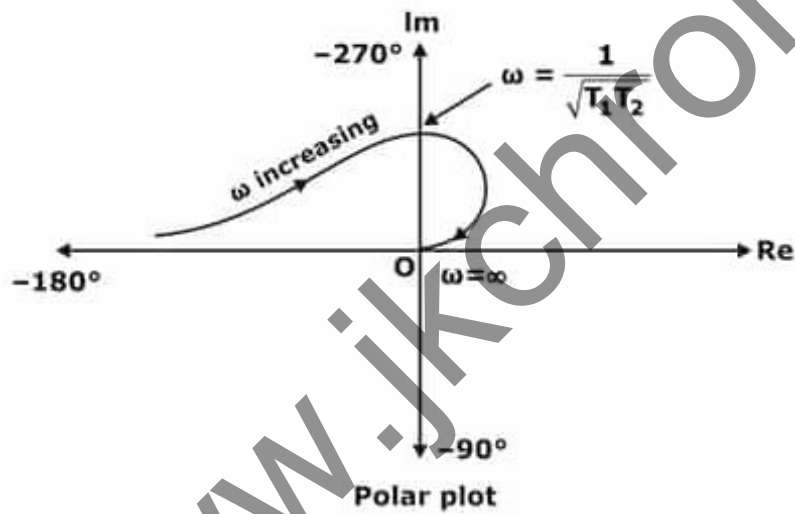
- Type 2 System

$$G(s) = \frac{K}{s^2(1+sT_1)}$$



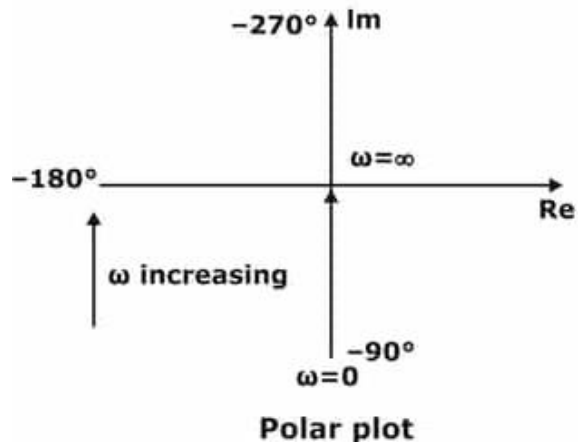
Introduction of Additional Pole

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$$



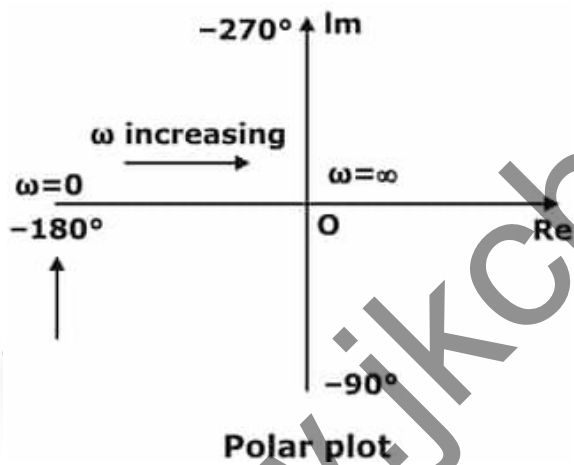
Polar Plot for

$$\frac{1}{s} = G(s)$$

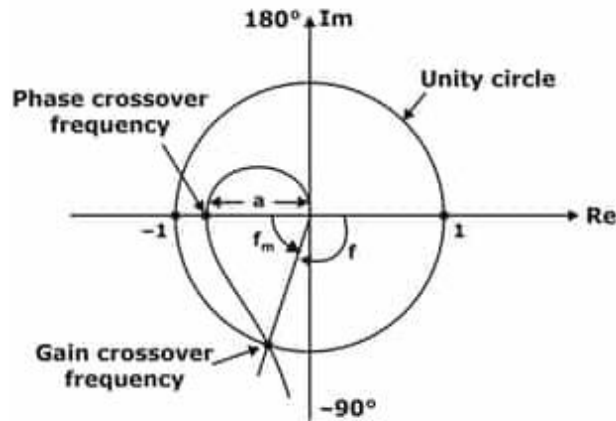


Polar Plot For

$$\frac{1}{s^2} = G(s)$$



Gain Margin and Phase Margin with Polar Plot



Polar plot to show GM and PM

- **Phase Crossover Frequency:** The frequency at which the polar plot crosses the -180° line is called the phase crossover frequency.
- **Gain Crossover Frequency:** The frequency at which the polar plot crosses the unit circle is called gain crossover frequency.
- **Gain Margin:** At phase crossover frequency, if the gain is 'a' then

$$\text{Gain margin} = -20 \log a$$

- **Phase Margin (φ_m):** At gain crossover frequency, if the phase is φ then phase margin

$$\varphi_m = 180 + \varphi$$

where, φ is positive from anti-clockwise direction.

Nyquist Stability Criterion

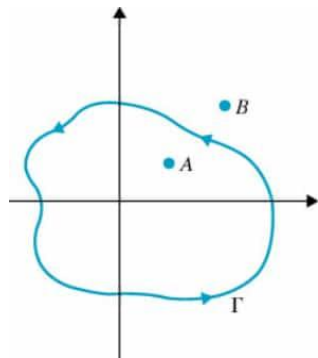
A stability test for time-invariant linear systems can also be derived in the frequency domain. It is known as the **Nyquist stability criterion**. It is based on the complex analysis result known as **Cauchy's principle** of argument. Nyquist criterion is used to identify the presence of roots of a characteristic equation of a control system in a specified region of the s-plane. The nyquist approach is the same as Routh-Hurwitz but, it differs with the following aspect:-

- The open loop transfer function " **$G(s)H(s)$** " is considered instead of closed loop characteristic equation " **$1+G(s)H(s) = 0$** ".
- Inspection of the graphical plot of $G(s)H(s)$ enables to get more than yes or no answer of Routh-Hurwitz method pertaining to the stability of control system.

- Nyquist plots display both amplitude and phase angle on a single plot, using frequency as a parameter in the plot.
- Nyquist plots have properties that allow you to see whether a system is stable or unstable. It will take some mathematical development to see that, but it's the most useful property of Nyquist plots.

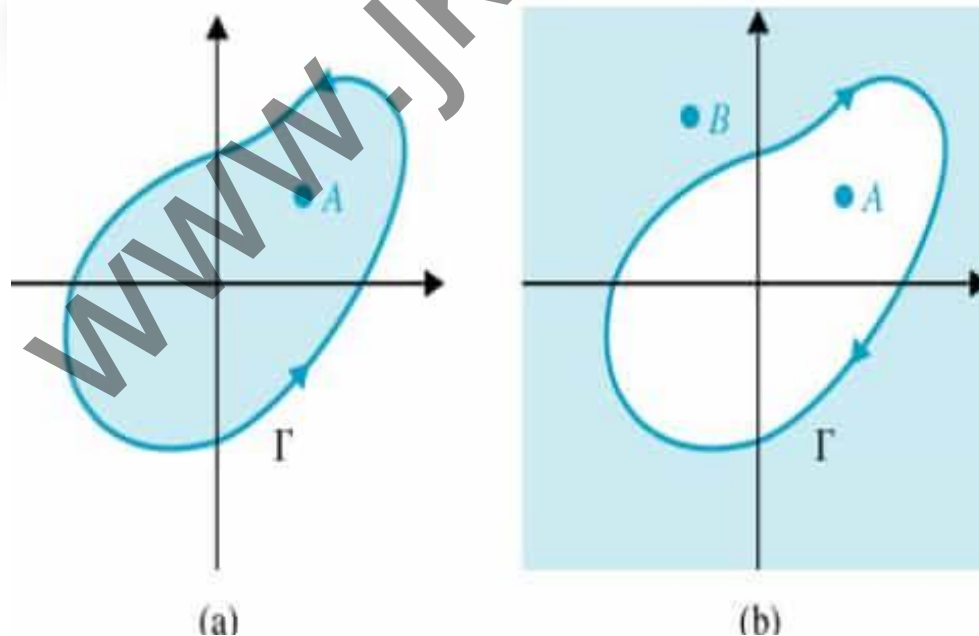
Concept of Encirclement & Enclosurement

Encirclement:



- point A is encircled in the counter in **Counter Clock Wise** by closed path, while point B is NOT encircled by closed path.

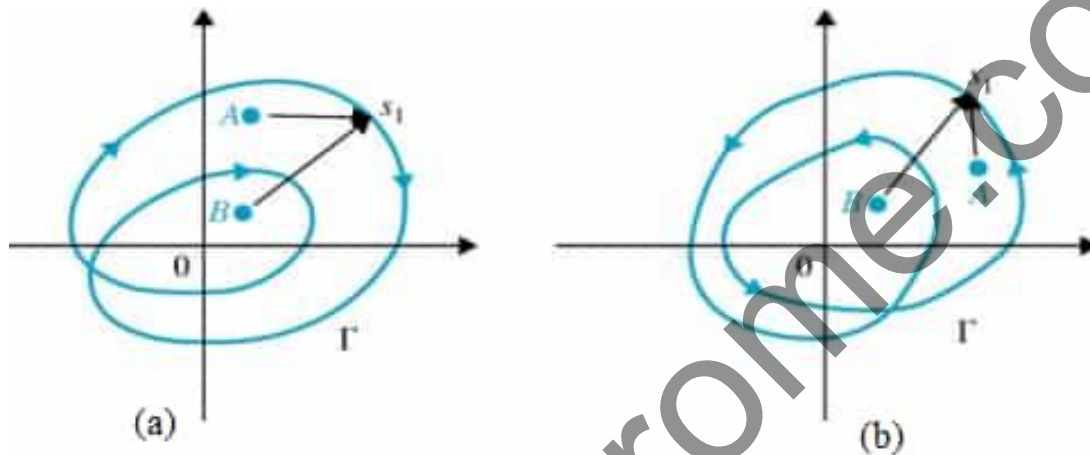
Enclosurement: A point or region is said to be enclosed by a closed path if the point or region lies to the right of the path when the path is traversed in any prescribed direction.



- In figure (a) point **A** is **not enclosed** by the path, while Point **B** is **enclosed** by the path.
- In figure (b) point **A** is **enclosed** by the path, while point **B** is **not enclosed** by the path.

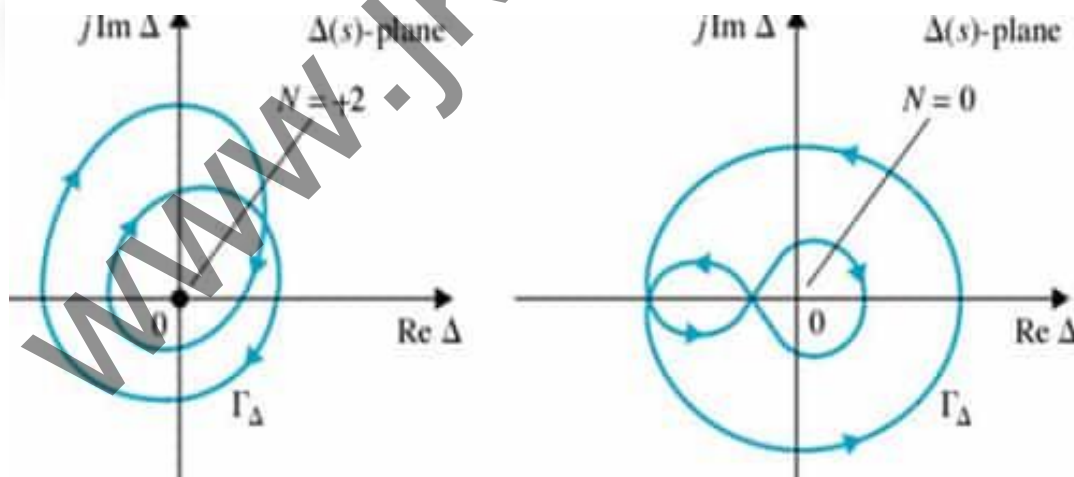
Nyquist Stability Criterion: Fundamentals

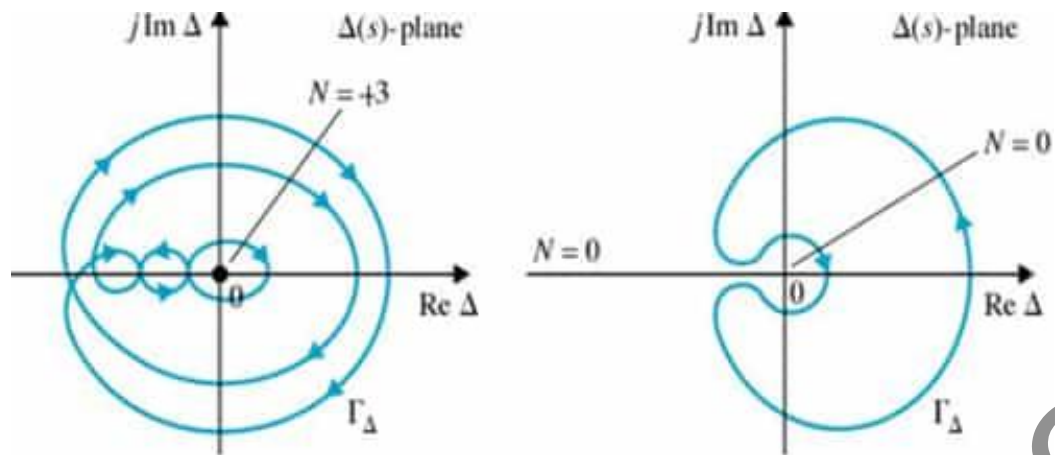
Consider the phasor from point A to S_1 of encirclement = N , the net angle traversed by the phasor = $2\pi N$ rad



- In figure (a) Point A of encirclement = +1; Point B of encirclement = +2.
- In figure (b) Point A of encirclement = -1; Point B of encirclement = -2.

Determination of N:





- For a SISO feedback system, the closed-loop transfer function is given by

$$M(s) = \frac{G(s)}{1 + H(s)G(s)}$$

- closed-loop system poles are obtained by solving the following equation

$$1 + G(s)H(s) = 0 = \Delta(s) \text{ represents the system characteristic equation.}$$

- In the following, we consider the complex function

$$D(s) = 1 + G(s)H(s)$$

- The zeros of $D(s)$ are the closed-loop poles of the transfer function. Here now we concluded that the poles of $D(s)$ are the zeros of $M(s)$.
- At the same time, the poles of $D(s)$ are the open-loop control system poles since they are contributed by the poles of $H(s)G(s)$, which can be considered as the open-loop control system transfer function—obtained when the feedback loop is open at some point.
- The Nyquist stability test is obtained by applying the Cauchy principle of argument to the complex function $D(s)$. First, we state Cauchy's principle of argument.

Cauchy's Principle of Argument

- Let $F(s)$ be an analytic function in a closed region of the complex s -plane, except at a finite number of points (namely, the poles of $F(s)$).
- It is also assumed that $F(s)$ is analytic at every point on the contour. Then, as ' s ' travels around the contour in the s -plane in the clockwise direction,

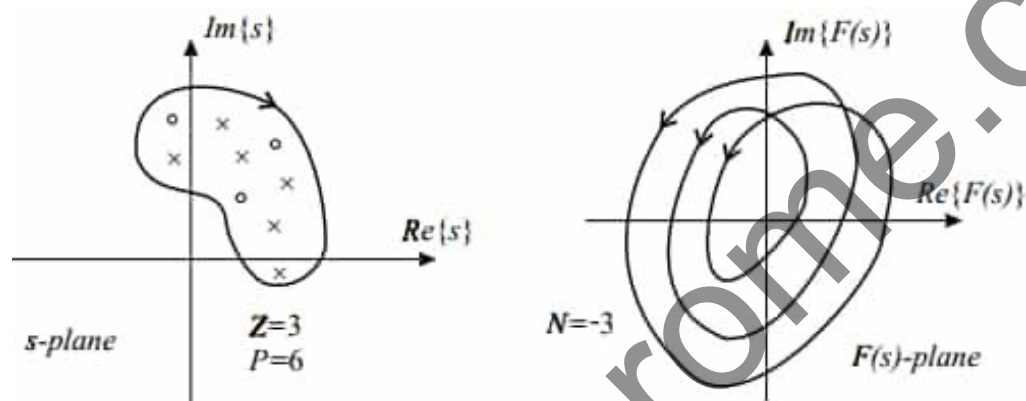
the function $F(s)$ encircles the origin in the $[\text{Re}\{F(s)\}, \text{Im}\{F(s)\}]$ -plane in the same direction N times, where N is given By

$$N = P - Z$$

where Z and P stand for the number of zeros and poles (including their multiplicities) of the function $F(s)$ inside the contour.

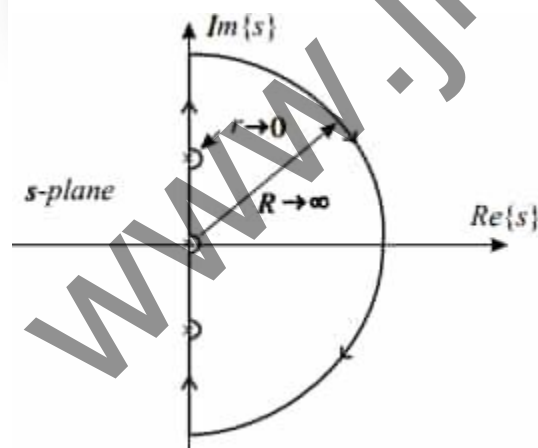
- The above result can be also written as

$$\text{Arg}\{F(s)\} = (Z - P)2\pi = 2N\pi$$



Nyquist Plot

The Nyquist plot is a polar plot of the function $D(s) = 1 + G(s)H(s)$ when travels around the contour



- The contour in the above figure covers the whole unstable half-plane of the complex plane s , $R \rightarrow \infty$.

- Since the function $D(s)$, according to Cauchy's principle of argument, must be analytic at every point on the contour, the poles $D(s)$ of on the imaginary axis must be encircled by infinitesimally small semicircles.

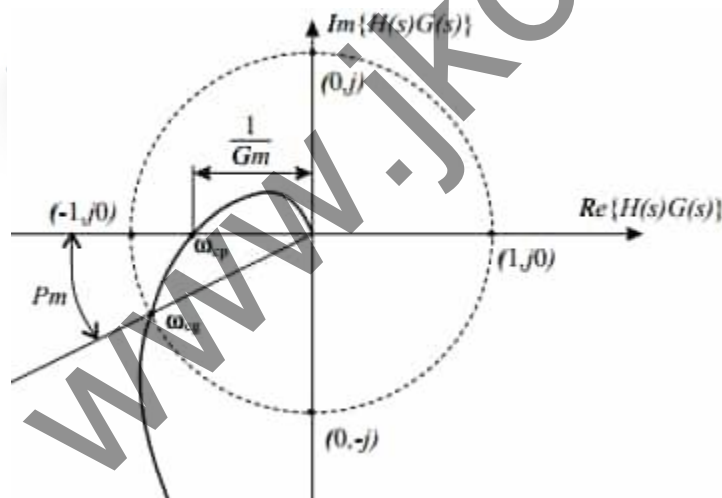
Nyquist Criterion:

- It states that the number of unstable closed-loop poles is equal to the number of unstable open-loop poles plus the number of encirclements of the origin of the Nyquist plot of the complex function $F(s)$.
- The above criterion can be slightly simplified if instead of plotting the function " $D(s) = 1+G(s)H(s)$ ", we plot only the function $G(s)H(s)$ and count encirclement of the Nyquist plot of $G(s)H(s)$ around the point $(-1+j0)$.
- The number of unstable closed-loop poles (Z) is equal to the number of unstable open-loop poles (P) plus the number of encirclements (N) of the point $(-1+j0)$ of the Nyquist plot of $G(s)H(s)$, that is

$$Z = P + N$$

Phase and Gain Stability Margins

Two important notions can be derived from the Nyquist diagram: phase and gain stability margins. The phase and gain stability margins are presented in the Figure below



- They give the degree of relative stability; in other words, they tell how far the given system is from the instability region. Their formal definitions are given by

$$Pm = 180^\circ + \arg \{G(j\omega_{cg})H(j\omega_{cg})\}$$

$$Gm [dB] = 20 \log \frac{1}{|G(j\omega_{cp})H(j\omega_{cp})|} [dB]$$

- where ω_{gc} and ω_{pc} stand for, respectively, the gain and phase crossover frequencies, which are obtained from the figure as

$$|G(j\omega_{cg})H(j\omega_{cg})| = 1 \Rightarrow \omega_{cg}$$

$$\arg \{G(j\omega_{cb})H(j\omega_{cb})\} = 180^\circ \Rightarrow \omega_{cb}$$

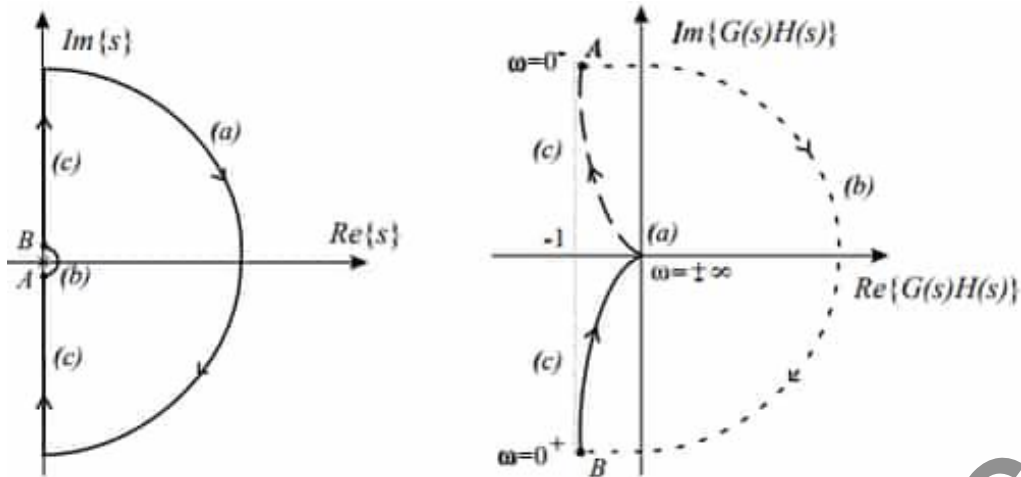
Example: Consider a control system represented by

$$G(s)H(s) = \frac{1}{s(s+1)}$$

Solution: Since this system has a pole at the origin, the contour in the s -plane should encircle it with a semicircle of an infinitesimally small radius. This contour has three parts (a), (b), and (c). Mappings for each of them are considered below.

- On this semicircle, the complex variable ' s ' is represented in the polar form by ' $s = Re^{j\psi}$ ' with $R \rightarrow \infty$, $-\pi/2 \leq \psi \leq \pi/2$. Substituting ' $s = Re^{j\psi}$ ' into $G(s)H(s)$, we easily see that $G(s)H(s) \rightarrow 0$. Thus, the huge semicircle from the s -plane maps into the origin in the $G(s)H(s)$ -plane.

Thus, the huge semicircle from the s -plane maps into the origin in the $G(s)H(s)$ -plane.



- On this semicircle, the complex variable is represented in the polar form by ' $s = re^{j\psi}$ ' with $r \rightarrow 0$, $-\pi/2 \leq \psi \leq \pi/2$. so that we have

$$G(s)H(s) \rightarrow \frac{1}{re^{j\psi}} \rightarrow \infty \times \arg(-\Phi)$$

Since ψ changes from ' $-\pi/2$ ' at point A to ' $\pi/2$ ' at point B, $\arg\{G(s)H(s)\}$ will change from " $\pi/2$ to $-\pi/2$ ". We conclude that the infinitesimally small semicircle at the origin in the s-plane is mapped into a semicircle of infinite radius in the $G(s)H(s)$ -plane.

- On this part of the contour, s takes pure imaginary values, i.e. $s = j\omega$ with ω changing from ' $-\infty$ to $+\infty$ '. Due to symmetry, it is sufficient to study only mapping along " $0^+ \leq \omega \leq +\infty$ ". We can find the real and imaginary parts of the function $G(j\omega)H(j\omega)$, which are given by

$$\text{Re}\{G(j\omega)H(j\omega)\} = \frac{-1}{\omega^2 + 1}$$

$$\text{Im}\{G(j\omega)H(j\omega)\} = \frac{-1}{\omega(\omega^2 + 1)}$$

From the above expressions, we see that neither the real nor the imaginary parts can be made zero, and hence the Nyquist plot has no points of intersection with the coordinate axis. For $\omega = 0^+$ we are at point B and since the plot at $\omega = +\infty$ will end up at the origin. Note that the vertical asymptote of the Nyquist plot is given by

$$\{\text{Re } G(j\omega)H(j\omega)\} = -1$$

- From the Nyquist diagram we see that $N=0$ and since there are no open-loop poles in the left half of the complex plane, i.e. $P=0$, we have $Z=0$ so that the corresponding closed-loop system has no unstable poles.

Controllers & Compensators

Compensation Technique

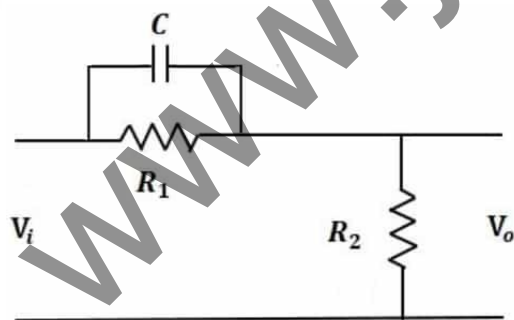
In Control Engineering we generally focused on methods used to analyze the performance of a feedback system with a given set of parameters. The results of such analysis frequently show that the performance of the feedback system is unacceptable for a given application because of such deficiencies as low Densensitivity, slow speed of response, or poor relative stability.

The process of modifying the system to improve performance is called Compensation.

The required device added in the control system to obtain the performance as per the desired specification is known as Compensator.

Phase lead Compensation:

A phase lead compensator improves the transient response of the system. As the name implies, this network provides a positive or leading phase shift of the output signal relative to the input signal at all frequencies. Lead-network parameters are usually selected to locate its singularities near the crossover frequency of the system being compensated. The positive phase shift of the network then improves the phase margin of the system.



A phase-lead compensator has a transfer function of the form:

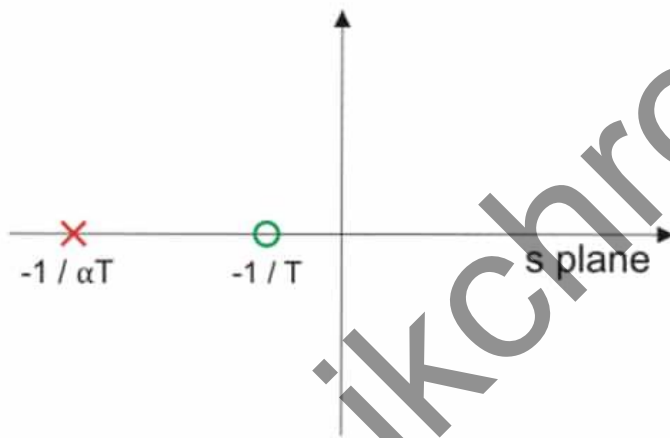
$$\frac{V_o}{V_i} = \frac{\alpha(1+\tau s)}{1+\alpha\tau s}$$

Where,

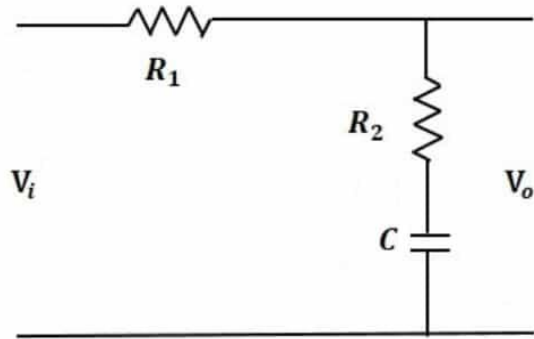
$$\tau = R_1 C$$

$$\alpha = \frac{R_2}{R_1 + R_2}$$

- A Lead Compensator is a high-pass filter.
- Zeros of the transfer function dominate in Phase Lead Compensation Technique.



Phase lag Compensation: Phase-lag systems are very common. These systems occur when an energy storage unit and an energy dissipator are combined. One example is the RC low pass. The phase-lag compensator has a negative phase angle and so is used to subtract phase from an uncompensated system.



A phase-lag compensator has a transfer function of the form:

$$\frac{V_o}{V_i} = \frac{1 + \tau s}{1 + \beta \tau s}$$

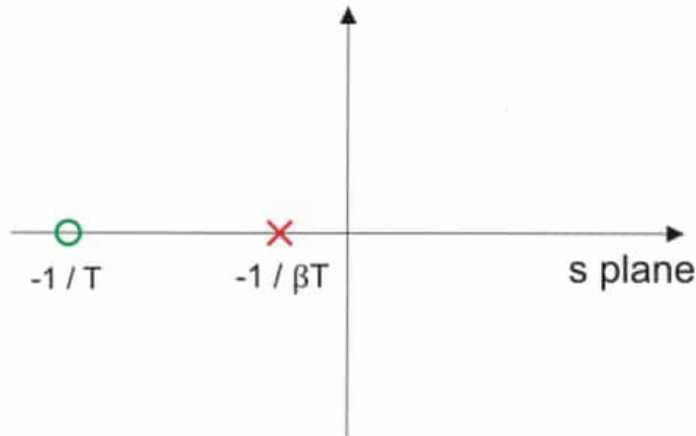
Where,

$$\tau = R_2 C$$

$$\beta = \frac{R_1 + R_2}{R_2}$$

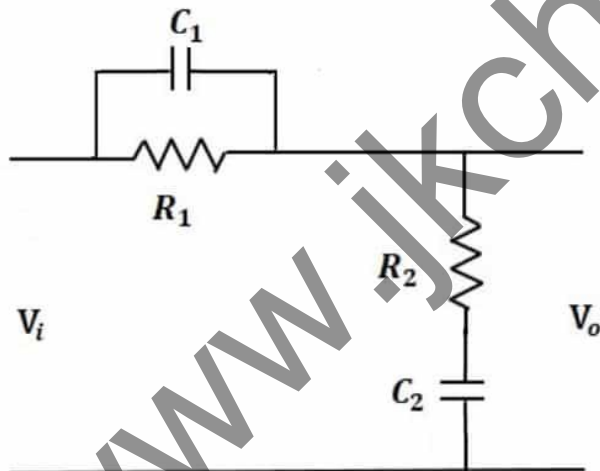
Because the phase-lag compensator adds a negative phase angle to a system, the phase lag is not a useful effect of the compensation and does not provide a direct means of improving the phase margin. The phase-lag compensator does, however, reduce the gain and so can be used to lower the crossover frequency. A consequence of this is that, usually, the phase margin of the system is higher at the lower frequency, the stability can be improved.

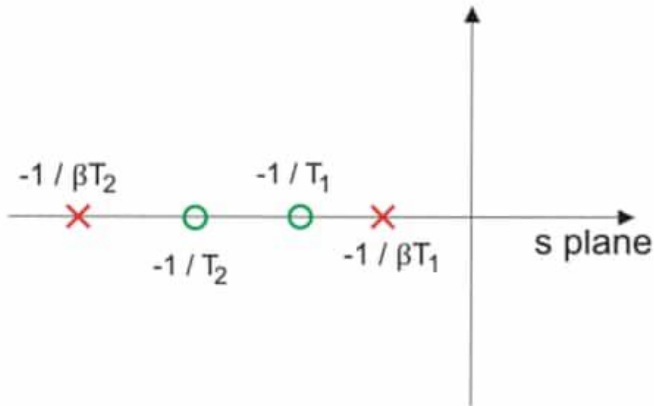
In other words, there should not be much change in the location of the dominant poles. The pole of a phase lag compensator lies extremely close to the origin and its zero is to the left of the pole and very near to it.



Phase lag lead Compensation: With single lag or lead compensation use, we may not be satisfied with design specifications. For an unstable uncompensated system, lead compensation provides a faster response but does not provide enough phase margin whereas lag compensation stabilizes the system but does not provide enough bandwidth. So we need multiple compensators in cascade i.e. phase lag-lead compensation.

It acts as a band stop filter





Controlling Action

- The controller (an analog/digital circuit, and software), is trying to keep the controlled variable such as temperature, liquid level, motor velocity, robot joint angle, at a certain value called the **setpoint (SP)**.
- A feedback control system does this by looking at the **error (E)** signal, which is the difference between where the controlled variable (called the **process variable (PV)**) is, and where it should be.
- Based upon the error signal, the controller decides the magnitude and the direction of the signal to the actuator.

The proportional (P), the integral (I), and the derivative (D), are all basic controllers.

Types of controllers: P, I, D, PI, PD, and PID controllers

- **Proportional Control**

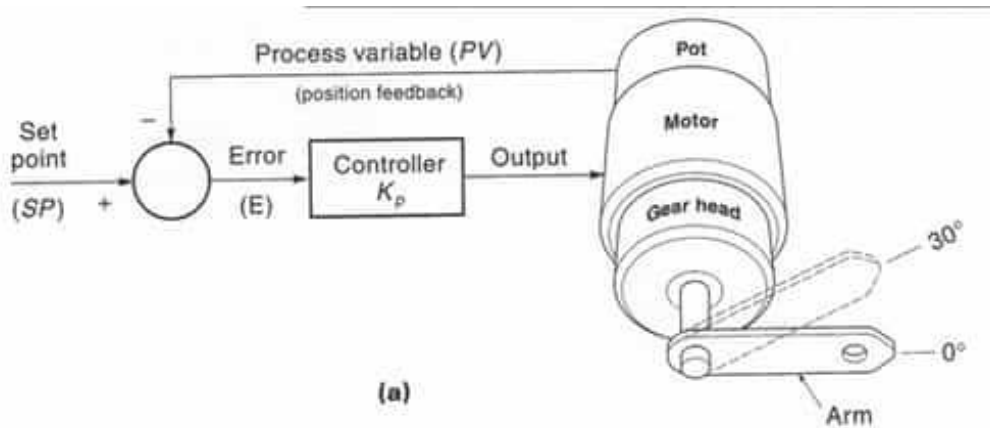
With proportional control, the actuator *applies a corrective force that is proportional to the amount of error*:

$$\text{Output}_p = K_p \times E$$

Output_p = system output due to proportional control

K_p = proportional constant for the system called **gain**

E = error, the difference between where the controlled variable should be and where it is. $E = SP - PV$.



One way to decrease the steady-state error is to increase the system gain (K_p), but high K_p can lead to instability problems.

Increasing K_p independently without limit is **not** a sound control strategy.

- **Integral Control**

The introduction of integral control in a control system can reduce the steady-state error to zero. Integral control *applies a restoring force that is proportional to the sum of all past errors, multiplied by time.*

$$\text{Output}_i = K_i \times \sum(E \times \Delta t)$$

Output_i = controller output due to integral control

K_i = integral gain constant (sometimes expressed as $1/T_i$)

$\sum(E \times \Delta t)$ = sum of all past errors (multiplied by time)

For a constant value of error $\sum(E \times \Delta t)$ will increase with time, causing the restoring force to get larger and larger.

Eventually, the restoring force will get large enough to overcome friction and move the controlled variable in a direction to eliminate the error.

- **Derivative Control**

One solution to the overshoot problem is to include derivative control. Derivative control 'applies the brakes,' slowing the controlled variable just before it reaches its destination.

Output_D = controller output due to derivative control

K_D = derivative gain constant

= error rate of change (slope of error curve)

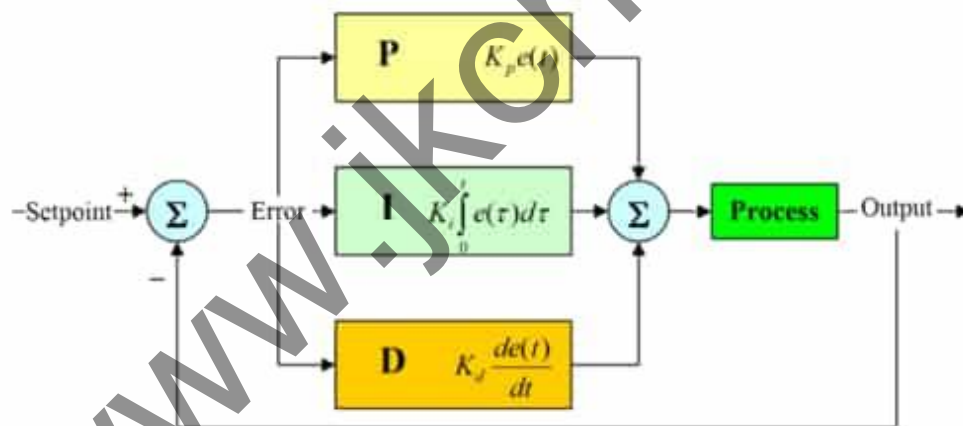
- **Combining P, I, and D controllers**

As proportional, integral and derivative controllers have their individual strengths and weaknesses, they are often combined so that their strengths are maximized, whilst minimizing their weaknesses. Many industrial controllers are a combination of P + I, or P + D, and are referred to as PI and PD controllers respectively.

- **PID control**

A proportional–integral–derivative controller (PID controller) is a generic control loop feedback mechanism (controller) widely used in industrial control systems.

A PID controller attempts to correct the error between a measured process variable and a desired setpoint by calculating and then outputting a corrective action that can adjust the process accordingly.



The foundation of the system is proportional control. Adding integral control provides a means to eliminate steady-state error, but increases overshoot. Derivative control increases stability by reducing the tendency to overshoot.

Simply adding together the three required control components generates the response of the PID system.

Output PID = output from PID controller

K_P = proportional control gain

K_I = integral control gain

K_D = derivative control gain

E = error (deviation from set point)

$\Sigma(E \times \Delta t)$ = sum of all past errors (area under the error/time curve)

$$\left(\frac{\Delta E}{\Delta t} \right) =$$

rate of change of error (slope of the error curve)

Equation is:

$$\text{Output}_{PID} = K_P E + \frac{1}{T_I} \int E dt + T_D \frac{dE}{dt}$$

When you are designing a PID controller for a given system, follow the steps shown below to obtain the desired response.

1. Obtain an open-loop response and determine what needs to be improved.
2. Add a proportional control to improve the rise time.
3. Add a derivative control to improve the overshoot.
4. Add an integral control to eliminate the steady-state error.
5. Adjust each of K_p , K_i , and K_d until you obtain a desired overall response.

The characteristics of P, I, and D controllers

- A proportional controller (K_p) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error.
- An integral control (K_i) will have the effect of eliminating the steady-state error, but it may make the transient response worse.
- A derivative control (K_d) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.

Parameter	P	PI	PD	PID
Time constant	Decrease	Increase	Decrease	Small Change(Decrease)
Rise Time	Decrease	Increase	Decrease	Small change(Decrease)
Peak Time	Decrease	Increase	Decrease	Small change(Decrease)
Overshoot	Increase	Decrease	Increase	Small change(Decrease)
Settling Time	Small Change	Increase	Decrease	Small change(Decrease)
Steady State Error	Decrease	Decrease	No Change	Decrease
Stability	Decrease(worse)	Decrease(worse)	Improve	Improve

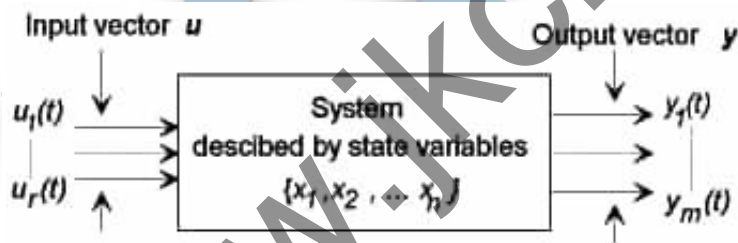
Comparison between P PI and PID controller.

State Space Analysis

The so-called state-space description provides the dynamics as a set of coupled first-order differential equations in a set of internal variables known as state variables, together with a set of algebraic equations that combine the state variables into physical output variables.

State: The state of a dynamic system refers to a minimum set of variables, known as state variables, that fully describe the system.

- A mathematical description of the system in terms of a minimum set of variables, $\mathbf{x}_i(\mathbf{t})$, $i = 1, \dots, n$, together with knowledge of those variables at an initial time t_0 , and the system inputs for time $t \geq t_0$, is sufficient to predict the future system state and outputs for all time $t > t_0$.



- The dynamic behavior of a state-determined system is completely characterized by the response of the set of n variables $x_i(t)$, where the number n is defined to be the order of the system.
- The system shown in the above figure has two **inputs** $u_1(t)$ and $u_2(t)$, and **four output** variables $y_1(t), \dots, y_4(t)$. If the system is state-determined, knowledge of its state variables ($x_1(t_0), x_2(t_0), \dots, x_n(t_0)$) at some initial time t_0 , and the inputs $u_1(t)$ and $u_2(t)$ for $t \geq t_0$ is sufficient to determine all future behavior of the system.

- The state variables are an internal description of the system which completely characterize the system state at any time t , and from which any output variables $y_i(t)$ may be computed.

The State Equations

The standard mathematical form of the system is expressed as a set of n coupled first-order ordinary differential equations, known as the state equations, in which the time derivative of each state variable is expressed in terms of the state variables $x_1(t), \dots, x_n(t)$ and the system inputs $u_1(t), \dots, u_r(t)$. In the general case the form of the n state equations is:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}, \mathbf{u}, t) \\ \dot{x}_2 &= f_2(\mathbf{x}, \mathbf{u}, t) \\ &\vdots \\ \dot{x}_n &= f_n(\mathbf{x}, \mathbf{u}, t) \end{aligned}$$

- where $\dot{\mathbf{x}} = d\mathbf{x}/dt$ and each of the functions $f_i(x, u, t)$, ($i = 1, \dots, n$) may be a general nonlinear, time-varying function of the state variables, the system inputs, and time.
- It is common to express the state equations in a vector form, in which the set of n state variables is written as a state vector $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$, and the set of r inputs is written as an input vector $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_r(t)]^T$. Each state variable is a time-varying component of the column vector $\mathbf{x}(t)$.
- In vector notation the set of n equations in Eqs. (1) may be written as

$$\dot{\mathbf{X}} = f(\mathbf{x}, \mathbf{u}, t)$$

where $f(x, u, t)$ is a vector function with n components $f_i(x, u, t)$.

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + \dots + b_{1r}u_r \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + \dots + b_{2r}u_r \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + \dots + b_{nr}u_r \end{aligned}$$

where the coefficients \mathbf{a}_{ij} and \mathbf{b}_{ij} are constants that describe the system.

The last equation can be written in matrix form as below

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1r} \\ b_{21} & & b_{2r} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

which may be summarized as:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

- Where state vector \mathbf{x} is a column vector of length n , the input vector \mathbf{u} is a column vector of length r , \mathbf{A} is an $n \times n$ square matrix of the constant coefficients a_{ij} , and \mathbf{B} is an $n \times r$ matrix of the coefficients b_{ij} that weight the inputs.

Output Equations

System output is defined to be any system variable of interest. An arbitrary output variable in a system of order n with r inputs may be written as

$$\mathbf{y}(t) = \mathbf{c}_1\mathbf{x}_1 + \mathbf{c}_2\mathbf{x}_2 + \dots + \mathbf{c}_n\mathbf{x}_n + \mathbf{d}_1\mathbf{u}_1 + \dots + \mathbf{d}_r\mathbf{u}_r$$

where the \mathbf{c}_i and \mathbf{d}_i are constants. If a total of m system variables are defined as outputs, then the output equation can also be obtained as State Equation in compact form

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

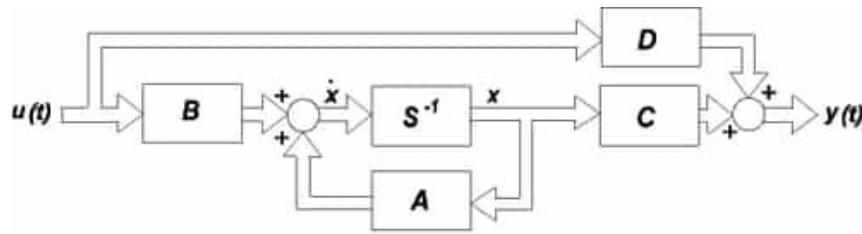
- where \mathbf{y} is a column vector of the output variables $y_i(t)$, \mathbf{C} is an $m \times n$ matrix of the constant coefficients c_{ij} that weight the state variables, and \mathbf{D} is an $m \times r$ matrix of the constant coefficients d_{ij} that weight the system inputs.
- For many physical systems, the matrix \mathbf{D} is the null matrix, and the output equation reduces to a simple weighted combination of the state variable

if $\mathbf{D} = \mathbf{0}$; then $\mathbf{Y} = \mathbf{C}\mathbf{x}$

Block Diagram Representation of Linear Systems Described by State Equations

A system of order n has n integrators in its block diagram. The derivatives of the state variables are the inputs to the integrator blocks, and each state equation expresses a derivative as a sum of weighted state variables and inputs. A

detailed block diagram representing a system of order n may be constructed directly from the state and output equations as follows:



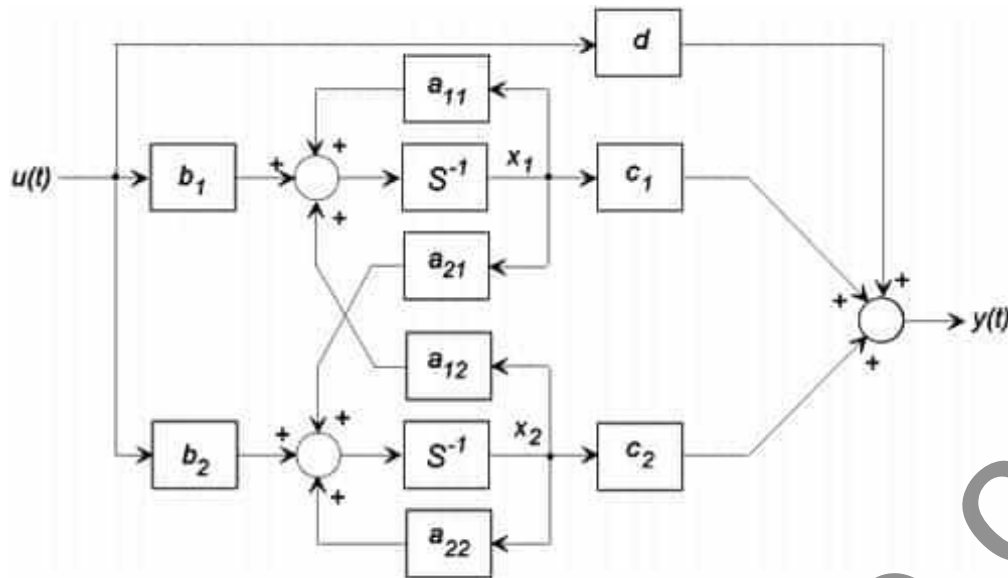
- Draw n integrator (S^{-1}) blocks, and assign a state variable to the output of each block.
- At the input to each block (which represents the derivative of its state variable) draw a summing element.
- Use the state equations to connect the state variables and inputs to the summing elements through scaling operator blocks.
- Expand the output equations and sum the state variables and inputs through a set of scaling operators to form the components of the output.

Example: Draw a block diagram for the general second-order, single-input single-output system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + du(t).$$

the block diagram shown below is drawn using the four steps described above



Transformation from Classical Form to State-Space Representation

Let the differential equation representing the system be of order n , and without loss of generality assume that the order of the polynomial operators on both sides is the same.

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0) Y(s) = (b_n s^n + b_{n-1} s^{n-1} + \dots + b_0) U(s)$$

- We may multiply both sides of the equation by s^{-n} to ensure that all differential operators have been eliminated

$$a_n + a_{n-1} s^{-1} + \dots + a_1 s^{-(n-1)} + a_0 s^{-n} Y(s) = b_n + b_{n-1} s^{-1} + \dots + b_1 s^{-(n-1)} + \dots + b_0 s^{-n} U(s)$$

from which the output may be specified in terms of a transfer function. If we define a dummy variable $Z(s)$, and split into two parts

$$Z(s) = \frac{1}{a_n + a_{n-1} s^{-1} + \dots + a_1 s^{-(n-1)} + a_0 s^{-n}} U(s)$$

$$Y(s) = (b_n + b_{n-1} s^{-1} + \dots + b_1 s^{-(n-1)} + b_0 s^{-n}) Z(s)$$

Eq. of $Z(s)$ can be solved for $U(s)$

$$U(s) = (a_n + a_{n-1} s^{-1} + \dots + a_1 s^{-(n-1)} + a_0 s^{-n}) X(s)$$

State-Space and Transfer Function

The state equation form

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

can be transformed into transfer function.

Taking the Laplace transform and neglect initial condition then

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$y(s) = CX(s) + DU(s)$$

$$\text{then } sX(s) - AX(s) = X(0) + BU(s)$$

By Neglecting Initial Conditions

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1} BU(s)$$

Then Put the value of X(s) for Y(s)...

$$\text{then } Y(s) = C(sI - A)^{-1} BU(s) + DU(s)$$

$$Y(s)/U(s) = G(s) = C(sI - A)^{-1} B + D$$

State-Transition Matrix

The state-transition matrix is defined as a matrix that satisfies the linear homogeneous state equation:

$$\frac{dx(t)}{dt} = Ax(t) \quad \dots\dots(i)$$

Let $\phi(t)$ be the $n \times n$ matrix that represents the state-transition matrix; then it must satisfy the equation:

$$\frac{d\phi(t)}{dt} = A\phi(t) \quad \text{(ii)}$$

Furthermore, let $x(0)$ denote the initial state at $t = 0$; then $\phi(t)$ is also defined by the matrix equation:

$$x(t) = \phi(t)x(0) \quad \text{..... (iii)}$$

which is the solution of the homogeneous state equation for $t \geq 0$. One way of determining $\phi(t)$ is by taking the Laplace transform on both sides of Eq. (i); we have

$$sX(s) - x(0) = AX(s) \quad \text{..... (iv)}$$

Solving for $X(s)$ from Eq. (v). we get

$$X(s) = (sI - A)^{-1}X(s) \quad \text{.... (v)}$$

where it is assumed that the matrix $(sI - A)$ is non-singular. Taking the inverse Laplace transform on both sides of Eq. (v) yields

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0) \quad t \geq 0 \text{.....(vi)}$$

By comparing Eq.(iv) with Eq. (v), the state-transition matrix is identified to be

$$\phi(t) = L^{-1}[(sI - A)^{-1}] \quad \text{..... (vii)}$$

An alternative way of solving the homogeneous state equation is to assume a solution, as in the classical method of solving linear differential equations. We let the solution to be

$$x(t) = e^{At}x(0)$$

for $t \geq 0$, where e^{At} represents the following power series of the matrix At , and

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots \text{.....(viii)}$$

Properties of State-Transition Matrices.

The state-transition matrix OM possesses the following properties:

1. $\phi(0) = I$ (the identity matrix)
2. $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_2 - t_1) \phi(t_1 - t_0) = \phi(t_2 - t_0)$ for any t_0, t_1, t_2
4. $[\phi(t)]^k = \phi(kt)$ for $k =$ positive integer

Controllability & Observability

Controllability: Controllability can be defined in order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable. A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state to any other state in a finite interval of time.

Condition for Controllability;

If the rank of $C_B = [B : AB : \dots : A^{n-1}B]$ is equal to n , then the system is controllable.

Observability: In order to see what is going on inside the system under observation, the system must be observable. A system is said to be observable at time t_0 if, with the system in state $X(t_0)$, it is possible to determine this state from the observation of the output over a finite interval of time.

Condition for Observability;

If the rank of $O_B = [C^T : A^T C^T : \dots : A^{n-1} C^T]$ is equal to n , then the system is said to be Observable.



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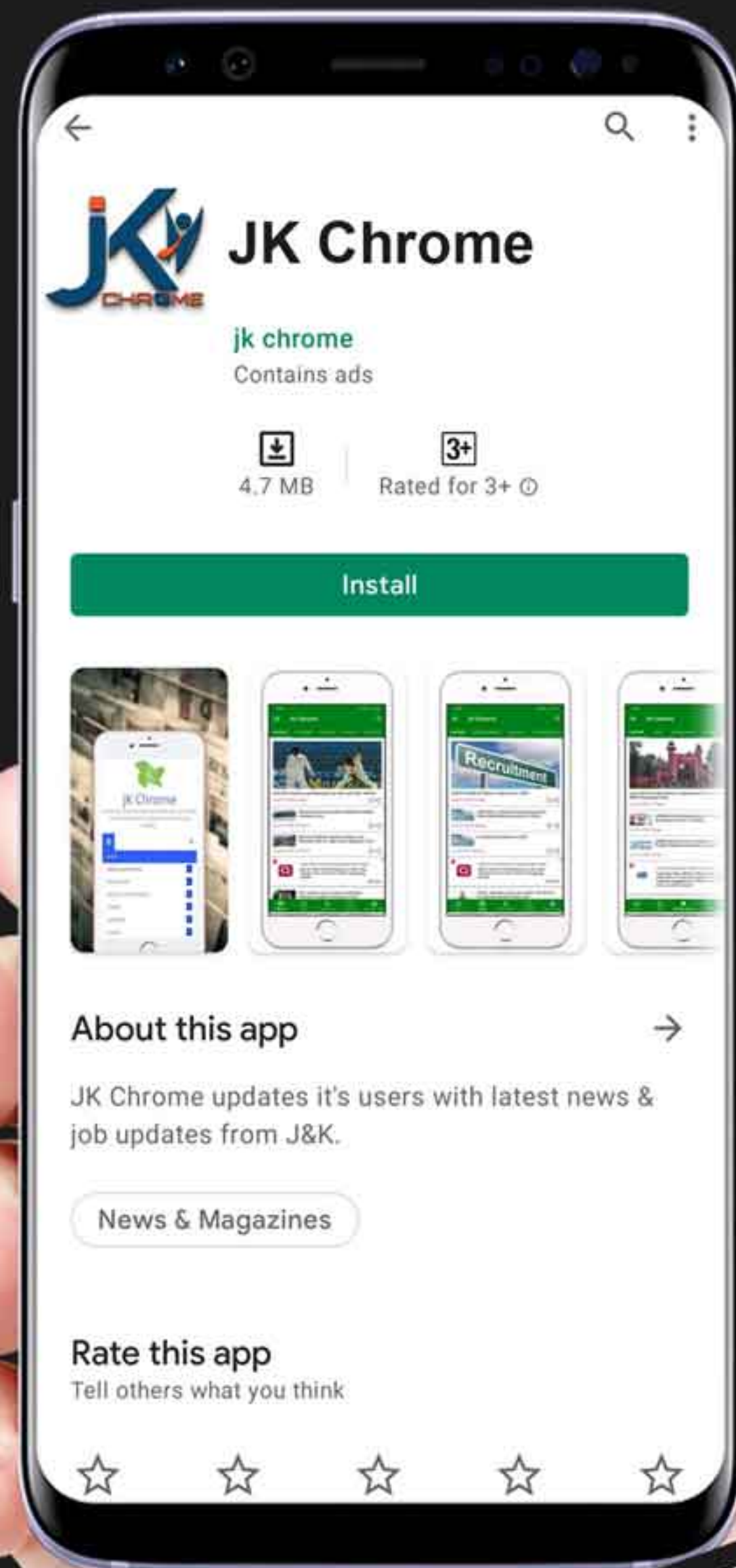
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